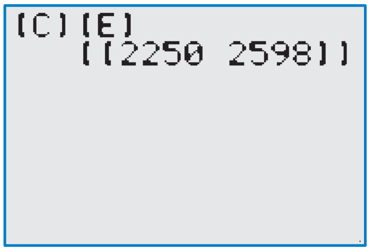
# 7 Linear Systems and Matrices



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Section 7.5, Example 12
Softball Team Expenses

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# 7.1 Solving Systems of Equations

## The Methods of Substitution and Graphing

So far in this text, most problems have involved either a function of one variable or a single equation in two variables. However, many problems in science, business, and engineering involve two or more equations in two or more variables. To solve such problems, you need to find solutions of **systems of equations.** Here is an example of a system of two equations in two unknowns, $x$ and $y$.

$$\begin{cases} 2x + y = 5 & \text{Equation 1} \\ 3x - 2y = 4 & \text{Equation 2} \end{cases}$$

A **solution** of this system is an ordered pair that satisfies each equation in the system. Finding the set of all such solutions is called **solving the system of equations.** For instance, the ordered pair (2, 1) is a solution of this system. To check this, you can substitute 2 for $x$ and 1 for $y$ in *each* equation.

*Check* (2, 1) *in Equation 1:*

$$2x + y = 5$$
$$2(2) + 1 \stackrel{?}{=} 5$$
$$5 = 5 \checkmark$$

*Check* (2, 1) *in Equation 2:*

$$3x - 2y = 4$$
$$3(2) - 2(1) \stackrel{?}{=} 4$$
$$4 = 4 \checkmark$$

In this section, you will study two ways to solve systems of equations, beginning with the **method of substitution.**

### The Method of Substitution

To use the **method of substitution** to solve a system of two equations in $x$ and $y$, perform the following steps.

1. Solve one of the equations for one variable in terms of the other.
2. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. Solve the equation obtained in Step 2.
4. Back-substitute the value(s) obtained in Step 3 into the expression obtained in Step 1 to find the value(s) of the other variable.
5. Check that each solution satisfies *both* of the original equations.

When using the **method of graphing,** note that the solution of the system corresponds to the **point(s) of intersection** of the graphs.

### The Method of Graphing

To use the **method of graphing** to solve a system of two equations in $x$ and $y$, perform the following steps.

1. Solve both equations for $y$ in terms of $x$.
2. Use a graphing utility to graph both equations in the same viewing window.
3. Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the point(s) of intersection of the graphs.
4. Check that each solution satisfies *both* of the original equations.

### *What you should learn*

- Use the methods of substitution and graphing to solve systems of equations in two variables.
- Use systems of equations to model and solve real-life problems.

### *Why you should learn it*

You can use systems of equations in situations in which the variables must satisfy two or more conditions. For instance, Exercise 88 on page 478 shows how to use a system of equations to compare two models for estimating the number of board feet in a 16-foot log.

### Study Tip

When using the method of substitution, it does not matter which variable you choose to solve for first. Whether you solve for $y$ first or $x$ first, you will obtain the same solution. When making your choice, you should choose the variable and equation that are easier to work with.

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## Example 1 Solving a System of Equations

Solve the system of equations.

$$\begin{cases} x + y = 4 & \text{Equation 1} \\ x - y = 2 & \text{Equation 2} \end{cases}$$

### Algebraic Solution

Begin by solving for $y$ in Equation 1.

$$y = 4 - x \qquad \text{Solve for } y \text{ in Equation 1.}$$

Next, substitute this expression for $y$ into Equation 2 and solve the resulting single-variable equation for $x$.

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ x - (4 - x) &= 2 && \text{Substitute } 4 - x \text{ for } y. \\ x - 4 + x &= 2 && \text{Distributive Property} \\ 2x - 4 &= 2 && \text{Combine like terms.} \\ 2x &= 6 && \text{Add 4 to each side.} \\ x &= 3 && \text{Divide each side by 2.} \end{aligned}$$

Finally, you can solve for $y$ by *back-substituting* $x = 3$ into the equation $y = 4 - x$ to obtain

$$\begin{aligned} y &= 4 - x && \text{Write revised Equation 1.} \\ y &= 4 - 3 && \text{Substitute 3 for } x. \\ y &= 1. && \text{Solve for } y. \end{aligned}$$

The solution is the ordered pair

$(3, 1)$.

Check this as follows.

*Check* $(3, 1)$ *in Equation 1:*

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\overset{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1. ✓} \end{aligned}$$

*Check* $(3, 1)$ *in Equation 2:*

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\overset{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2. ✓} \end{aligned}$$

Because $(3, 1)$ satisfies both equations in the system, it is a solution of the system of equations.

✓CHECKPOINT Now try Exercise 19.

### Graphical Solution

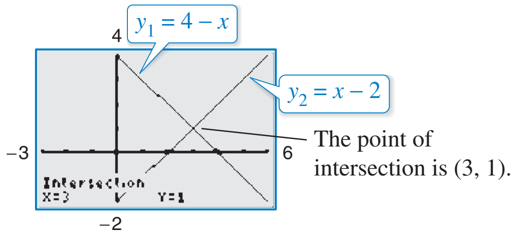
Begin by solving both equations for $y$. Then use a graphing utility to graph the equations

$$y_1 = 4 - x$$

and

$$y_2 = x - 2$$

in the same viewing window. Use the *intersect* feature (see Figure 7.1) to approximate the point of intersection of the graphs.



The point of intersection is (3, 1).

**Figure 7.1**

Check that $(3, 1)$ is the exact solution as follows.

*Check* $(3, 1)$ *in Equation 1:*

$$\begin{aligned} x + y &= 4 && \text{Write Equation 1.} \\ 3 + 1 &\overset{?}{=} 4 && \text{Substitute for } x \text{ and } y. \\ 4 &= 4 && \text{Solution checks in Equation 1. ✓} \end{aligned}$$

*Check* $(3, 1)$ *in Equation 2:*

$$\begin{aligned} x - y &= 2 && \text{Write Equation 2.} \\ 3 - 1 &\overset{?}{=} 2 && \text{Substitute for } x \text{ and } y. \\ 2 &= 2 && \text{Solution checks in Equation 2. ✓} \end{aligned}$$

Because $(3, 1)$ satisfies both equations in the system, it is a solution of the system of equations.

In the algebraic solution of Example 1, note that the term *back-substitution* implies that you work *backwards*. First you solve for one of the variables, and then you substitute that value *back* into one of the equations in the system to find the value of the other variable.

## Example 2 Solving a System by Substitution

A total of $12,000 is invested in two funds paying 9% and 11% simple interest. The yearly interest is $1180. How much is invested at each rate?

### Solution

*Verbal Model:* 9% fund + 11% fund = Total investment

9% fund + 11% fund = Total interest

*Labels:* Amount in 9% fund = $x$ (dollars)

Interest for 9% fund = $0.09x$ (dollars)

Amount in 11% fund = $y$ (dollars)

Interest for 11% fund = $0.11y$ (dollars)

Total investment = $12,000 (dollars)

Total interest = $1180 (dollars)

*System:*

$$\begin{cases} x + y = 12{,}000 & \text{Equation 1} \\ 0.09x + 0.11y = 1180 & \text{Equation 2} \end{cases}$$

To begin, it is convenient to multiply each side of Equation 2 by 100. This eliminates the need to work with decimals.

$$9x + 11y = 118{,}000 \qquad \text{Revised Equation 2}$$

To solve this system, you can solve for $x$ in Equation 1.

$$x = 12{,}000 - y \qquad \text{Revised Equation 1}$$

Next, substitute this expression for $x$ into revised Equation 2 and solve the resulting equation for $y$.

$$\begin{aligned} 9x + 11y &= 118{,}000 && \text{Write revised Equation 2.} \\ 9(12{,}000 - y) + 11y &= 118{,}000 && \text{Substitute } 12{,}000 - y \text{ for } x. \\ 108{,}000 - 9y + 11y &= 118{,}000 && \text{Distributive Property} \\ 2y &= 10{,}000 && \text{Combine like terms.} \\ y &= 5000 && \text{Divide each side by 2.} \end{aligned}$$

Finally, back-substitute the value $y = 5000$ to solve for $x$.

$$\begin{aligned} x &= 12{,}000 - y && \text{Write revised Equation 1.} \\ x &= 12{,}000 - 5000 && \text{Substitute 5000 for } y. \\ x &= 7000 && \text{Simplify.} \end{aligned}$$

The solution is

$$(7000, 5000).$$

So, $7000 is invested at 9% and $5000 is invested at 11% to yield yearly interest of $1180. Check this in the original system.

✓CHECKPOINT Now try Exercise 29.

The equations in Examples 1 and 2 are linear. Substitution and graphing can also be used to solve systems in which one or both of the equations are nonlinear.

### Technology Tip

Remember that a good way to check the answers you obtain in this section is to use a graphing utility. For instance, enter the two equations in Example 2

$$y_1 = 12{,}000 - x$$

$$y_2 = \frac{1180 - 0.09x}{0.11}$$

and find an appropriate viewing window that shows where the lines intersect. Then use the *intersect* feature or the *zoom* and *trace* features to find the point of intersection.

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## Example 3 Substitution: Two-Solution Case

Solve the system of equations.

$$\begin{cases} x^2 + 4x - y = 7 & \text{Equation 1} \\ 2x - y = -1 & \text{Equation 2} \end{cases}$$

### Algebraic Solution

Begin by solving for $y$ in Equation 2 to obtain

$$y = 2x + 1. \qquad \text{Solve for } y \text{ in Equation 2.}$$

Next, substitute this expression for $y$ into Equation 1 and solve for $x$.

$$x^2 + 4x - y = 7 \qquad \text{Write Equation 1.}$$

$$x^2 + 4x - (2x + 1) = 7 \qquad \text{Substitute } 2x + 1 \text{ for } y.$$

$$x^2 + 4x - 2x - 1 = 7 \qquad \text{Distributive Property}$$

$$x^2 + 2x - 8 = 0 \qquad \text{Write in general form.}$$

$$(x + 4)(x - 2) = 0 \qquad \text{Factor.}$$

$$x + 4 = 0 \Longrightarrow x = -4 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 2 = 0 \Longrightarrow x = 2 \qquad \text{Set 2nd factor equal to 0.}$$

Back-substituting these values of $x$ into revised Equation 2 produces

$$y = 2(-4) + 1 = -7 \quad \text{and} \quad y = 2(2) + 1 = 5.$$

So, the solutions are

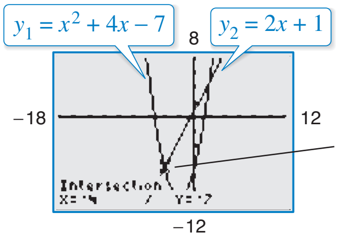
$$(-4, -7) \quad \text{and} \quad (2, 5).$$

Check these in the original system.

CHECKPOINT Now try Exercise 33.

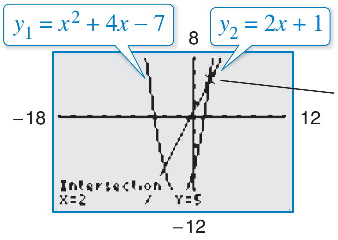
### Graphical Solution

Solve each equation for $y$ and use a graphing utility to graph the equations in the same viewing window.



One of the points of intersection is (−4, −7).

**Figure 7.2**



One of the points of intersection is (2, 5).

**Figure 7.3**

From Figures 7.2 and 7.3, the solutions are $(-4, -7)$ and $(2, 5)$. Check these in the original system.

## Example 4 Substitution: No-Solution Case

Solve the system of equations.

$$\begin{cases} -x + y = 4 & \text{Equation 1} \\ x^2 + y = 3 & \text{Equation 2} \end{cases}$$

### Solution

Begin by solving for $y$ in Equation 1 to obtain $y = x + 4$. Next, substitute this expression for $y$ into Equation 2 and solve for $x$.

$$x^2 + y = 3 \qquad \text{Write Equation 2.}$$

$$x^2 + (x + 4) = 3 \qquad \text{Substitute } x + 4 \text{ for } y.$$

$$x^2 + x + 1 = 0 \qquad \text{Simplify.}$$

$$x = \frac{-1 \pm \sqrt{3}i}{2} \qquad \text{Quadratic Formula}$$

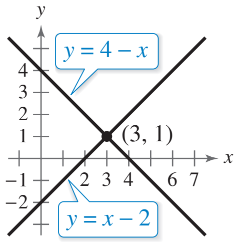
Because this yields two complex values, the equation $x^2 + x + 1 = 0$ has no *real* solution. So, the original system of equations has no *real* solution.

CHECKPOINT Now try Exercise 35.

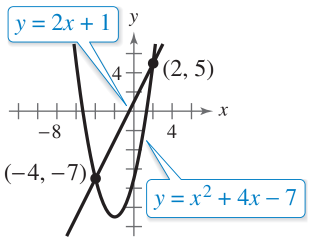
### Explore the Concept

Graph the system of equations in Example 4. Do the graphs of the equations intersect? Why or why not?

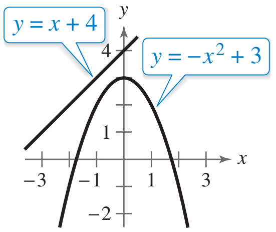
From Examples 1, 3, and 4, you can see that a system of two equations in two unknowns can have exactly one solution, more than one solution, or no solution. For instance, in Figure 7.4, the two equations in Example 1 graph as two lines with a *single point* of intersection. The two equations in Example 3 graph as a parabola and a line with *two points* of intersection, as shown in Figure 7.5. The two equations in Example 4 graph as a line and a parabola that have *no points* of intersection, as shown in Figure 7.6.



***One Intersection Point***
**Figure 7.4**



***Two Intersection Points***
**Figure 7.5**



***No Intersection Points***
**Figure 7.6**

Example 5 shows the benefit of a graphical approach to solving systems of equations in two variables. Notice what happens when you try only the substitution method in Example 5. You obtain the equation $x + \ln x = 1$. It is difficult to solve this equation for $x$ using standard algebraic techniques. In such cases, a graphical approach to solving systems of equations is more convenient.

## Example 5 Solving a System of Equations Graphically

Solve the system of equations.

$$\begin{cases} y = \ln x & \text{Equation 1} \\ x + y = 1 & \text{Equation 2} \end{cases}$$

### Solution

From the graphs of these equations, it is clear that there is only one point of intersection. Use the *intersect* feature of a graphing utility to approximate the solution point as $(1, 0)$, as shown in Figure 7.7.

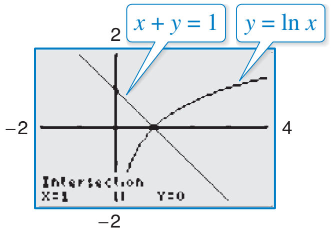


**Figure 7.7**

You can confirm this by substituting $(1, 0)$ into *both* equations.

*Check* $(1, 0)$ *in Equation 1:*

$y = \ln x$ Write Equation 1.

$0 = \ln 1$ Equation 1 checks. ✓

*Check* $(1, 0)$ *in Equation 2:*

$x + y = 1$ Write Equation 2.

$1 + 0 = 1$ Equation 2 checks. ✓

✓CHECKPOINT Now try Exercise 55.

## Application

The total cost $C$ of producing $x$ units of a product typically has two components: the initial cost and the cost per unit. When enough units have been sold so that the total revenue $R$ equals the total cost $C$, the sales are said to have reached the **break-even point.** You will find that the break-even point corresponds to the point of intersection of the cost and revenue curves.

### Example 6 Break-Even Analysis

A small business invests \$10,000 in equipment to produce a new soft drink. Each bottle of the soft drink costs \$0.65 to produce and is sold for \$1.20. How many bottles must be sold before the business breaks even?

#### Solution

The total cost of producing $x$ bottles is

$$\boxed{\text{Total cost}} = \boxed{\text{Cost per bottle}} \cdot \boxed{\text{Number of bottles}} + \boxed{\text{Initial cost}}$$

$$C = 0.65x + 10{,}000. \qquad \text{Equation 1}$$

The revenue obtained by selling $x$ bottles is

$$\boxed{\text{Total revenue}} = \boxed{\text{Price per bottle}} \cdot \boxed{\text{Number of bottles}}$$

$$R = 1.20x. \qquad \text{Equation 2}$$

Because the break-even point occurs when $R = C$, you have

$$C = 1.20x$$

and the system of equations to solve is

$$\begin{cases} C = 0.65x + 10{,}000 \\ C = 1.20x \end{cases}.$$

Now you can solve by substitution.

$$C = 0.65x + 10{,}000 \qquad \text{Write Equation 1.}$$

$$1.20x = 0.65x + 10{,}000 \qquad \text{Substitute } 1.20x \text{ for } C.$$

$$0.55x = 10{,}000 \qquad \text{Subtract } 0.65x \text{ from each side.}$$

$$x = \frac{10{,}000}{0.55} \qquad \text{Divide each side by 0.55.}$$

$$x \approx 18{,}182 \text{ bottles.} \qquad \text{Use a calculator.}$$

Note in Figure 7.8 that revenue less than the break-even point corresponds to an overall loss, whereas revenue greater than the break-even point corresponds to a profit. Verify the break-even point using the *intersect* feature or the *zoom* and *trace* features of a graphing utility.

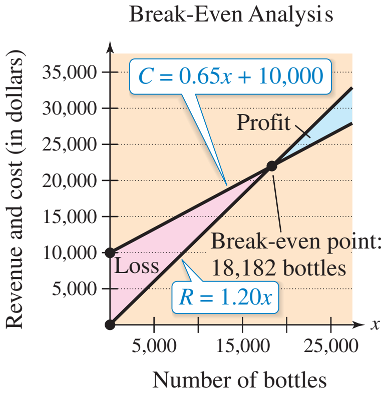


**Figure 7.8**

CHECKPOINT Now try Exercise 83.

Another way to view the solution in Example 6 is to consider the profit function

$$P = R - C. \qquad \text{Profit function}$$

The break-even point occurs when the profit is 0, which is the same as saying that $R = C$.

## 7.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1–4, fill in the blank(s).**

**1.** A set of two or more equations in two or more unknowns is called a ________ of ________ .

**2.** A ________ of a system of equations is an ordered pair that satisfies each equation in the system.

**3.** The first step in solving a system of two equations in $x$ and $y$ by the method of ________ is to solve one of the equations for one variable in terms of the other.

**4.** A point of intersection of the graphs of the equations of a system is a ________ of the system.

**5.** What is the point of intersection of the graphs of the cost and revenue functions called?

**6.** The graphs of the equations of a system do not intersect. What can you conclude about the system?

### Procedures and Problem Solving

**Checking Solutions** **In Exercises 7–10, determine whether each ordered pair is a solution of the system of equations.**

**7.** $\begin{cases} 4x - y = 1 \\ 6x + y = -6 \end{cases}$ (a) $(0, -3)$ (b) $(-1, -5)$ (c) $\left(-\frac{3}{2}, 3\right)$ (d) $\left(-\frac{1}{2}, -3\right)$

**8.** $\begin{cases} 4x^2 + y = 3 \\ -x - y = 11 \end{cases}$ (a) $(2, -13)$ (b) $(-2, -9)$ (c) $\left(-\frac{3}{2}, 6\right)$ (d) $\left(-\frac{7}{4}, -\frac{37}{4}\right)$

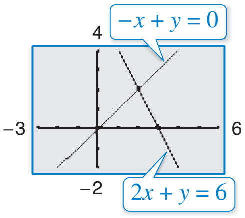
**9.** $\begin{cases} y = -2e^x \\ 3x - y = 2 \end{cases}$ (a) $(-2, 0)$ (b) $(0, -2)$ (c) $(0, -3)$ (d) $(-1, -5)$

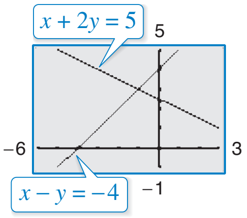
**10.** $\begin{cases} -\log_{10} x + 3 = y \\ \frac{1}{9}x + y = \frac{28}{9} \end{cases}$ (a) $(100, 1)$ (b) $(10, 2)$ (c) $(1, 3)$ (d) $(1, 1)$

**Solving a System by Substitution** **In Exercises 11–18, solve the system by the method of substitution. Check your solution graphically.**

**11.** $\begin{cases} 2x + y = 6 \\ -x + y = 0 \end{cases}$

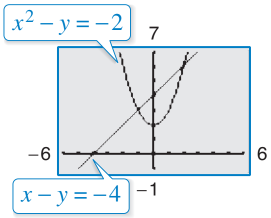
**12.** $\begin{cases} x - y = -4 \\ x + 2y = 5 \end{cases}$

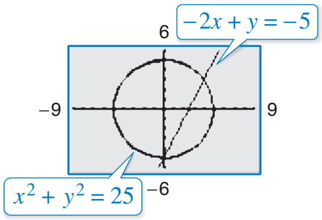




**13.** $\begin{cases} x - y = -4 \\ x^2 - y = -2 \end{cases}$

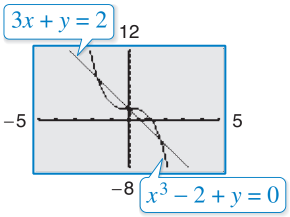
**14.** $\begin{cases} -2x + y = -5 \\ x^2 + y^2 = 25 \end{cases}$

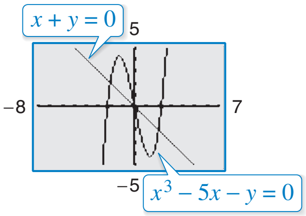




**15.** $\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$

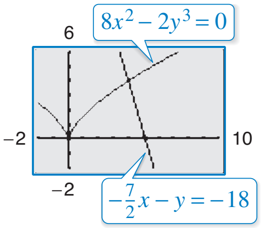
**16.** $\begin{cases} x + y = 0 \\ x^3 - 5x - y = 0 \end{cases}$

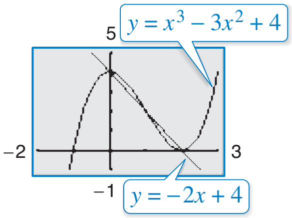




**17.** $\begin{cases} -\frac{7}{2}x - y = -18 \\ 8x^2 - 2y^3 = 0 \end{cases}$

**18.** $\begin{cases} y = x^3 - 3x^2 + 4 \\ y = -2x + 4 \end{cases}$





**Solving a System of Equations** **In Exercises 19–28, solve the system by the method of substitution. Use a graphing utility to verify your results.**

✓ **19.** $\begin{cases} x - y = 0 \\ 5x - 3y = 10 \end{cases}$

**20.** $\begin{cases} x + 2y = 1 \\ 5x - 4y = -23 \end{cases}$

**21.** $\begin{cases} 2x - y + 2 = 0 \\ 4x + y - 5 = 0 \end{cases}$

**22.** $\begin{cases} 6x - 3y - 4 = 0 \\ x + 2y - 4 = 0 \end{cases}$

**23.** $\begin{cases} 1.5x + 0.8y = 2.3 \\ 0.3x - 0.2y = 0.1 \end{cases}$

**24.** $\begin{cases} 0.5x + 3.2y = 9.0 \\ 0.2x - 1.6y = -3.6 \end{cases}$

**25.** $\begin{cases} \frac{1}{5}x + \frac{1}{2}y = 8 \\ x + y = 20 \end{cases}$

**26.** $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 10 \\ \frac{3}{4}x - y = 4 \end{cases}$

**27.** $\begin{cases} -\frac{5}{3}x + y = 5 \\ -5x + 3y = 6 \end{cases}$ **28.** $\begin{cases} -\frac{2}{3}x + y = 2 \\ 2x - 3y = 6 \end{cases}$

**Solving a System by Substitution** **In Exercises 29–32, you are given the yearly interest earned from a total of \$18,000 invested in two funds paying the given rates of simple interest. Write and solve a system of equations to find the amount invested at each rate.**

| | *Yearly Interest* | *Rate 1* | *Rate 2* |
|---|---|---|---|
| ✓ **29.** | \$1000 | 4% | 6% |
| **30.** | \$1140 | 5% | 7% |
| **31.** | \$1542 | 7.6% | 8.8% |
| **32.** | \$684 | 2.75% | 4.25% |

**Solving a System with a Nonlinear Equation** **In Exercises 33–38, solve the system by the method of substitution. Use a graphing utility to verify your results.**

✓ **33.** $\begin{cases} x^2 - 2x + y = 8 \\ x - y = -2 \end{cases}$ **34.** $\begin{cases} 2x^2 - 2x - y = 14 \\ 2x - y = -2 \end{cases}$

✓ **35.** $\begin{cases} 2x^2 - y = 1 \\ x - y = 2 \end{cases}$ **36.** $\begin{cases} 2x^2 + y = 3 \\ x + y = 4 \end{cases}$

**37.** $\begin{cases} x^3 - y = 0 \\ x - y = 0 \end{cases}$ **38.** $\begin{cases} y = -x \\ y = x^3 + 3x^2 + 2x \end{cases}$

**Solving a System of Equations Graphically** **In Exercises 39–46, solve the system graphically. Verify your solutions algebraically.**

**39.** $\begin{cases} -x + 2y = 2 \\ 3x + y = 15 \end{cases}$ **40.** $\begin{cases} x + y = 0 \\ 3x - 2y = 10 \end{cases}$

**41.** $\begin{cases} x - 3y = -2 \\ 5x + 3y = 17 \end{cases}$ **42.** $\begin{cases} -x + 2y = 1 \\ x - y = 2 \end{cases}$

**43.** $\begin{cases} x^2 + y = 1 \\ x + y = 2 \end{cases}$ **44.** $\begin{cases} x^2 - y = 4 \\ x - y = 2 \end{cases}$

**45.** $\begin{cases} -x + y = 3 \\ x^2 + y^2 - 6x - 27 = 0 \end{cases}$

**46.** $\begin{cases} y^2 - 4x + 11 = 0 \\ -\frac{1}{2}x + y = -\frac{1}{2} \end{cases}$

**Solving a System of Equations Graphically** **In Exercises 47–60, use a graphing utility to approximate all points of intersection of the graphs of equations in the system. Round your results to three decimal places. Verify your solutions by checking them in the original system.**

**47.** $\begin{cases} 7x + 8y = 24 \\ x - 8y = 8 \end{cases}$ **48.** $\begin{cases} x - y = 0 \\ 5x - 2y = 6 \end{cases}$

**49.** $\begin{cases} x - y^2 = -1 \\ x - y = 5 \end{cases}$ **50.** $\begin{cases} x - y^2 = -2 \\ x - 2y = 6 \end{cases}$

**51.** $\begin{cases} x^2 + y^2 = 8 \\ y = x^2 \end{cases}$ **52.** $\begin{cases} x^2 + y^2 = 25 \\ (x - 8)^2 + y^2 = 41 \end{cases}$

**53.** $\begin{cases} y = e^x \\ x - y + 1 = 0 \end{cases}$ **54.** $\begin{cases} y = -4e^{-x} \\ y + 3x + 8 = 0 \end{cases}$

✓ **55.** $\begin{cases} x + 2y = 8 \\ y = 2 + \ln x \end{cases}$

**56.** $\begin{cases} y = -2 + \ln(x - 1) \\ 3y + 2x = 9 \end{cases}$

**57.** $\begin{cases} y = \sqrt{x} + 4 \\ y = 2x + 1 \end{cases}$ **58.** $\begin{cases} x - y = 3 \\ \sqrt{x} - y = 1 \end{cases}$

**59.** $\begin{cases} x^2 + y^2 = 169 \\ x^2 - 8y = 104 \end{cases}$ **60.** $\begin{cases} x^2 + y^2 = 4 \\ 2x^2 - y = 2 \end{cases}$

**Choosing a Solution Method** **In Exercises 61–74, solve the system graphically or algebraically. Explain your choice of method.**

**61.** $\begin{cases} 2x - y = 0 \\ x^2 - y = -1 \end{cases}$ **62.** $\begin{cases} x + y = 4 \\ x^2 + y = 2 \end{cases}$

**63.** $\begin{cases} 3x - 7y = -6 \\ x^2 - y^2 = 4 \end{cases}$ **64.** $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$

**65.** $\begin{cases} x^2 + y^2 = 1 \\ x + y = 4 \end{cases}$ **66.** $\begin{cases} x^2 + y^2 = 4 \\ x - y = 5 \end{cases}$

**67.** $\begin{cases} y = 2x + 1 \\ y = \sqrt{x + 2} \end{cases}$ **68.** $\begin{cases} y = 2x - 1 \\ y = \sqrt{x + 1} \end{cases}$

**69.** $\begin{cases} y - e^{-x} = 1 \\ y - \ln x = 3 \end{cases}$ **70.** $\begin{cases} 2\ln x + y = 4 \\ e^x - y = 0 \end{cases}$

**71.** $\begin{cases} y = x^3 - 2x^2 + 1 \\ y = 1 - x^2 \end{cases}$ **72.** $\begin{cases} y = x^3 - 2x^2 + x - 1 \\ y = -x^2 + 3x - 1 \end{cases}$

**73.** $\begin{cases} xy - 1 = 0 \\ 2x - 4y + 7 = 0 \end{cases}$

**74.** $\begin{cases} xy - 2 = 0 \\ 3x - 2y + 4 = 0 \end{cases}$

**Break-Even Analysis** **In Exercises 75–78, use a graphing utility to graph the cost and revenue functions in the same viewing window. Find the sales $x$ necessary to break even ($R = C$) and the corresponding revenue $R$ obtained by selling $x$ units. (Round to the nearest whole unit.)**

| | *Cost* | *Revenue* |
|---|---|---|
| **75.** | $C = 8650x + 250{,}000$ | $R = 9950x$ |
| **76.** | $C = 2.65x + 350{,}000$ | $R = 4.15x$ |
| **77.** | $C = 5.5\sqrt{x} + 10{,}000$ | $R = 3.29x$ |
| **78.** | $C = 7.8\sqrt{x} + 18{,}500$ | $R = 12.84x$ |

**Geometry** **In Exercises 79 and 80, find the dimensions of the rectangle meeting the specified conditions.**

**79.** The perimeter is 30 meters and the length is 3 meters greater than the width.

**80.** The perimeter is 280 centimeters and the width is 20 centimeters less than the length.

**81. Marketing Research** The daily DVD rentals of a newly released animated film and a newly released horror film from a movie rental store can be modeled by the equations

$$\begin{cases} N = 360 - 24x & \text{Animated film} \\ N = 24 + 18x & \text{Horror film} \end{cases}$$

where $N$ is the number of DVDs rented and $x$ represents the week, with $x = 1$ corresponding to the first week of release.

(a) Use the *table* feature of a graphing utility to find the numbers of rentals of each movie for each of the first 12 weeks of release.

(b) Use the results of part (a) to determine the solution to the system of equations.

(c) Solve the system of equations algebraically.

(d) Compare your results from parts (b) and (c).

(e) Interpret the results in the context of the situation.

**82. Economics** You want to buy either a wood pellet stove or an electric furnace. The pellet stove costs $2160 and produces heat at a cost of $15.15 per 1 million Btu (British thermal units). The electric furnace costs $1250 and produces heat at a cost of $33.25 per 1 million Btu.

(a) Write a function for the total cost $y$ of buying the pellet stove and producing $x$ million Btu of heat.

(b) Write a function for the total cost $y$ of buying the electric furnace and producing $x$ million Btu of heat.

(c) Use a graphing utility to graph and solve the system of equations formed by the two cost functions.

(d) Solve the system of equations algebraically.

(e) Interpret the results in the context of the situation.

✓ **83. Break-Even Analysis** A small software company invests $16,000 to produce a software package that will sell for $55.95. Each unit can be produced for $9.45.

(a) Write the cost and revenue functions for $x$ units produced and sold.

(b) Use a graphing utility to graph the cost and revenue functions in the same viewing window. Use the graph to approximate the number of units that must be sold to break even, and verify the result algebraically.

**84. Professional Sales** You are offered two jobs selling college textbooks. One company offers an annual salary of $30,000 plus a year-end bonus of 1% of your total sales. The other company offers an annual salary of $25,000 plus a year-end bonus of 2% of your total sales. How much would you have to sell in a year to make the second offer the better offer?

**85. Geometry** What are the dimensions of a rectangular tract of land with a perimeter of 40 miles and an area of 96 square miles?

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**86. Geometry** What are the dimensions of an isosceles right triangle with a two-inch hypotenuse and an area of 1 square inch?

**87. Finance** You are deciding how to invest a total of $20,000 in two funds paying 5.5% and 7.5% simple interest. You want to earn a total of $1300 in interest from the investments each year.

(a) Write a system of equations in which one equation represents the total amount invested and the other equation represents the $1300 yearly interest. Let $x$ and $y$ represent the amounts invested at 5.5% and 7.5%, respectively.

(b) Use a graphing utility to graph the two equations in the same viewing window.

(c) How much of the $20,000 should you invest at 5.5% to earn $1300 in interest per year? Explain your reasoning.

**88. *Why you should learn it*** *(p. 470)* You are offered two different rules for estimating the number of board feet in a 16-foot log. (A board foot is a unit of measure for lumber equal to a board 1 foot square and 1 inch thick.) One rule is the Doyle Log Rule modeled by

$$V = (D - 4)^2, \quad 5 \le D \le 40$$

where $D$ is the diameter (in inches) of the log and $V$ is its volume in (board feet). The other rule is the Scribner Log Rule modeled by

$$V = 0.79D^2 - 2D - 4, \quad 5 \le D \le 40.$$

(a) Use a graphing utility to graph the two log rules in the same viewing window.

(b) For what diameter do the two rules agree?

(c) You are selling large logs by the board foot. Which rule would you use? Explain your reasoning.

**89. Algebraic-Graphical-Numerical** The populations (in thousands) of Arizona $A$ and Indiana $I$ from 2000 through 2008 can by modeled by the system

$$\begin{cases} A = 171.9t + 5118 & \text{Arizona} \\ I = 35.7t + 6080 & \text{Indiana} \end{cases}$$

where $t$ is the year, with $t = 0$ corresponding to 2000. (Source: U.S. Census Bureau)

(a) Record in a table the populations given by the models for the two states in the years 2000 through 2008.

(b) According to the table, in what year(s) was the population of Arizona greater than that of Indiana?

(c) Use a graphing utility to graph the models in the same viewing window. Estimate the point of intersection of the models.

(d) Find the point of intersection algebraically.

(e) Summarize your findings of parts (b) through (d).

**90. MODELING DATA**

The table shows the yearly revenues (in millions of dollars) of the online travel companies Expedia and Priceline.com from 2004 through 2008. (Sources: Expedia; Priceline.com)

| Year | Expedia | Priceline.com |
|---|---|---|
| 2004 | 1843 | 914 |
| 2005 | 2120 | 963 |
| 2006 | 2238 | 1123 |
| 2007 | 2665 | 1391 |
| 2008 | 2937 | 1885 |

(a) Use the *regression* feature of a graphing utility to find a linear model for the yearly revenue $E$ of Expedia and a quadratic model for the yearly revenue $P$ of Priceline.com. Let $x$ represent the year, with $x = 4$ corresponding to 2004.

(b) Use the graphing utility to graph the models with the original data in the same viewing window.

(c) Use the graph in part (b) to approximate the first year when the revenues of Priceline.com will be greater than the revenues of Expedia.

(d) Algebraically approximate the first year when the revenues of Priceline.com will be greater than the revenues of Expedia.

(e) Compare your results from parts (c) and (d).

## Conclusions

**True or False?** **In Exercises 91 and 92, determine whether the statement is true or false. Justify your answer.**

**91.** In order to solve a system of equations by substitution, you must always solve for $y$ in one of the two equations and then back-substitute.

**92.** If a system consists of a parabola and a circle, then it can have at most two solutions.

**93. Think About It** When solving a system of equations by substitution, how do you recognize that the system has no solution?

**94. Exploration** Find the equations of lines whose graphs intersect the graph of the parabola $y = x^2$ at (a) two points, (b) one point, and (c) no points. (There are many correct answers.)

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**95. Exploration** Create systems of two linear equations in two variables that have (a) no solution, (b) one distinct solution, and (c) infinitely many solutions. (There are many correct answers.)

**96. Exploration** Create a system of linear equations in two variables that has the solution $(2, -1)$ as its only solution. (There are many correct answers.)

**97. Exploration** Consider the system of equations.

$$\begin{cases} y = b^x \\ y = x^b \end{cases}$$

(a) Use a graphing utility to graph the system of equations for $b = 2$ and $b = 4$.

(b) For a fixed value of $b > 1$, make a conjecture about the number of points of intersection of the graphs in part (a).

**98. CAPSTONE** Consider the system of equations

$$\begin{cases} ax + by = c \\ dx + cy = f \end{cases}.$$

(a) Find values of $a$, $b$, $c$, $d$, $e$, and $f$ such that the system has one distinct solution. (There is more than one correct answer.)

(b) Explain how to solve the system in part (a) by the method of substitution and graphically.

(c) Write a brief paragraph describing any advantages of the method of substitution over the graphical method of solving a system of equations.

## Cumulative Mixed Review

**Finding the Slope-Intercept Form** **In Exercises 99–104, write an equation of the line passing through the two points. Use the slope-intercept form, if possible. If not possible, explain why.**

**99.** $(-2, 7), (5, 5)$

**100.** $(3, 4), (10, 6)$

**101.** $(6, 3), (10, 3)$

**102.** $(4, -2), (4, 5)$

**103.** $\left(\frac{3}{5}, 0\right), (4, 6)$

**104.** $\left(-\frac{7}{3}, 8\right), \left(\frac{5}{2}, \frac{1}{2}\right)$

**Finding the Domain and Asymptotes of a Function** **In Exercises 105–110, find the domain of the function and identify any horizontal or vertical asymptotes.**

**105.** $f(x) = \dfrac{5}{x - 6}$

**106.** $f(x) = \dfrac{2x - 7}{3x + 2}$

**107.** $f(x) = \dfrac{x^2 + 2}{x^2 - 16}$

**108.** $f(x) = 3 - \dfrac{2}{x^2}$

**109.** $f(x) = \dfrac{x + 1}{x^2 + 1}$

**110.** $f(x) = \dfrac{x - 4}{x^2 + 16}$

# 7.2 Systems of Linear Equations in Two Variables

## The Method of Elimination

In Section 7.1, you studied two methods for solving a system of equations: substitution and graphing. Now you will study the **method of elimination** to solve a system of linear equations in two variables. The key step in this method is to obtain, for one of the variables, coefficients that differ only in sign so that *adding* the equations eliminates the variable.

$$\begin{aligned} 3x + 5y &= 7 && \text{Equation 1} \\ -3x - 2y &= -1 && \text{Equation 2} \\ \hline 3y &= 6 && \text{Add equations.} \end{aligned}$$

Note that by adding the two equations, you eliminate the $x$-terms and obtain a single equation in $y$. Solving this equation for $y$ produces

$$y = 2$$

which you can then back-substitute into one of the original equations to solve for $x$.

### What you should learn

- Use the method of elimination to solve systems of linear equations in two variables.
- Graphically interpret the number of solutions of a system of linear equations in two variables.
- Use systems of linear equations to model and solve real-life problems.

### Why you should learn it

You can use systems of linear equations to model many business applications. For instance, Exercise 84 on page 487 shows how to use a system of linear equations to determine number of running shoes sold.

### Example 1 Solving a System by Elimination

Solve the system of linear equations.

$$\begin{cases} 3x + 2y = 4 & \text{Equation 1} \\ 5x - 2y = 8 & \text{Equation 2} \end{cases}$$

### Solution

Because the coefficients of $y$ differ only in sign, you can eliminate the $y$-terms by adding the two equations.

$$\begin{aligned} 3x + 2y &= 4 && \text{Write Equation 1.} \\ 5x - 2y &= 8 && \text{Write Equation 2.} \\ \hline 8x &= 12 && \text{Add equations.} \\ x &= \tfrac{3}{2} && \text{Solve for } x. \end{aligned}$$

So, $x = \frac{3}{2}$. By back-substituting into Equation 1, you can solve for $y$.

$$\begin{aligned} 3x + 2y &= 4 && \text{Write Equation 1.} \\ 3\left(\tfrac{3}{2}\right) + 2y &= 4 && \text{Substitute } \tfrac{3}{2} \text{ for } x. \\ y &= -\tfrac{1}{4} && \text{Solve for } y. \end{aligned}$$

The solution is

$$\left(\tfrac{3}{2}, -\tfrac{1}{4}\right).$$

You can check the solution *algebraically* by substituting into the original system, or graphically as shown in Section 7.1.

### Check

$$\begin{aligned} 3\left(\tfrac{3}{2}\right) + 2\left(-\tfrac{1}{4}\right) &\overset{?}{=} 4 && \text{Substitute into Equation 1.} \\ \tfrac{9}{2} - \tfrac{1}{2} &= 4 && \text{Equation 1 checks. } \checkmark \\ 5\left(\tfrac{3}{2}\right) - 2\left(-\tfrac{1}{4}\right) &\overset{?}{=} 8 && \text{Substitute into Equation 2.} \\ \tfrac{15}{2} + \tfrac{1}{2} &= 8 && \text{Equation 2 checks. } \checkmark \end{aligned}$$

✓CHECKPOINT Now try Exercise 13.

### Explore the Concept

Use the method of substitution to solve the system given in Example 1. Which method is easier?

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### The Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in $x$ and $y$, perform the following steps.

1. Obtain coefficients for $x$ (or $y$) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.
2. Add the equations to eliminate one variable; solve the resulting equation.
3. Back-substitute the value obtained in Step 2 into either of the original equations and solve for the other variable.
4. Check your solution in both of the original equations.

## Example 2 Solving a System by Elimination

Solve the system of linear equations.

$$\begin{cases} 5x + 3y = 9 & \text{Equation 1} \\ 2x - 4y = 14 & \text{Equation 2} \end{cases}$$

### Algebraic Solution

You can obtain coefficients of $y$ that differ only in sign by multiplying Equation 1 by 4 and multiplying Equation 2 by 3.

$$\begin{aligned} 5x + 3y &= 9 & \Rightarrow \quad 20x + 12y &= 36 & &\text{Multiply Equation 1 by 4.} \\ \underline{2x - 4y} &\underline{= 14} & \Rightarrow \quad \underline{6x - 12y} &\underline{= 42} & &\text{Multiply Equation 2 by 3.} \\ & & 26x \qquad &= 78 & &\text{Add equations.} \end{aligned}$$

From this equation, you can see that $x = 3$. By back-substituting this value of $x$ into Equation 2, you can solve for $y$.

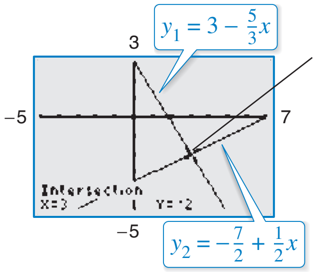
$$\begin{aligned} 2x - 4y &= 14 & &\text{Write Equation 2.} \\ 2(3) - 4y &= 14 & &\text{Substitute 3 for } x. \\ -4y &= 8 & &\text{Combine like terms.} \\ y &= -2 & &\text{Solve for } y. \end{aligned}$$

The solution is $(3, -2)$. You can check the solution algebraically by substituting into the original system.

**CHECKPOINT** Now try Exercise 15.

### Graphical Solution

Solve each equation for $y$ and use a graphing utility to graph the equations in the same viewing window.



The point of intersection is (3, −2).

**Figure 7.9**

From Figure 7.9, the solution is

$(3, -2)$.

Check this in the original system.

In Example 2, the two systems of linear equations (the original system and the system obtained by multiplying by constants)

$$\begin{cases} 5x + 3y = 9 \\ 2x - 4y = 14 \end{cases} \quad \text{and} \quad \begin{cases} 20x + 12y = 36 \\ 6x - 12y = 42 \end{cases}$$

are called **equivalent systems** because they have precisely the same solution set. Each of the following operations can be performed on a system of linear equations to produce an equivalent system.

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one of the equations to any other equation in the system.

## Graphical Interpretation of Two-Variable Systems

It is possible for any system of equations to have exactly one solution, two or more solutions, or no solution. If a system of *linear* equations has two different solutions, then it must have an *infinite* number of solutions. To see why this is true, consider the following graphical interpretations of a system of two linear equations in two variables.

### Graphical Interpretations of Solutions

For a system of two linear equations in two variables, the number of solutions is one of the following.

| *Number of Solutions* | *Graphical Interpretation* |
|---|---|
| **1.** Exactly one solution | The two lines intersect at one point. |
| **2.** Infinitely many solutions | The two lines are coincident (identical). |
| **3.** No solution | The two lines are parallel. |

A system of linear equations is **consistent** when it has at least one solution. It is **inconsistent** when it has no solution.

### Explore the Concept

Rewrite each system of equations in slope-intercept form and use a graphing utility to graph each system. What is the relationship between the slopes of the two lines and the number of points of intersection?

**a.** $\begin{cases} y = 5x + 1 \\ y - x = -5 \end{cases}$

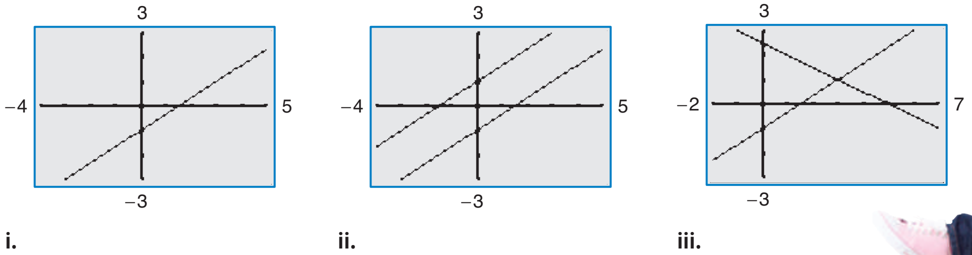
**b.** $\begin{cases} 3y = 4x - 1 \\ -8x + 2 = -6y \end{cases}$

**c.** $\begin{cases} 2y = -x + 3 \\ -4 = y + \frac{1}{2}x \end{cases}$

### Example 3 Recognizing Graphs of Linear Systems

Match each system of linear equations (a, b, c) with its graph (i, ii, iii) in Figure 7.10. Describe the number of solutions. Then state whether the system is consistent or inconsistent.

**a.** $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = 6 \end{cases}$ **b.** $\begin{cases} 2x - 3y = 3 \\ x + 2y = 5 \end{cases}$ **c.** $\begin{cases} 2x - 3y = 3 \\ -4x + 6y = -6 \end{cases}$



i. ii. iii.

**Figure 7.10**

### Solution

Begin by rewriting each system of equations in slope-intercept form.

System (a): $\begin{cases} y = \frac{2}{3}x - 1 \\ y = \frac{2}{3}x + 1 \end{cases}$ System (b): $\begin{cases} y = \frac{2}{3}x - 1 \\ y = -\frac{1}{2}x + \frac{5}{2} \end{cases}$ System (c): $\begin{cases} y = \frac{2}{3}x - 1 \\ y = \frac{2}{3}x - 1 \end{cases}$

**a.** The graph of system (a) is a pair of parallel lines (ii). The lines have no point of intersection, so the system has no solution. The system is inconsistent.

**b.** The graph of system (b) is a pair of intersecting lines (iii). The lines have one point of intersection, so the system has exactly one solution. The system is consistent.

**c.** The graph of system (c) is a pair of lines that coincide (i). The lines have infinitely many points of intersection, so the system has infinitely many solutions. The system is consistent.

✓CHECKPOINT Now try Exercises 23–26.

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In Examples 4 and 5, note how you can use the method of elimination to determine that a system of linear equations has no solution or infinitely many solutions.

## Example 4 The Method of Elimination: No-Solution Case

Solve the system of linear equations.

$$\begin{cases} x - 2y = 3 & \text{Equation 1} \\ -2x + 4y = 1 & \text{Equation 2} \end{cases}$$

### Algebraic Solution

To obtain coefficients that differ only in sign, multiply Equation 1 by 2.

$$\begin{array}{rcr} x - 2y = 3 & \Rightarrow & 2x - 4y = 6 \\ \underline{-2x + 4y = 1} & \Rightarrow & \underline{-2x + 4y = 1} \\ & & 0 = 7 \end{array}$$

By adding the equations, you obtain $0 = 7$. Because there are no values of $x$ and $y$ for which

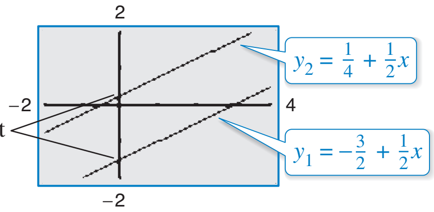
$$0 = 7 \qquad \text{False statement}$$

this is a false statement. So, you can conclude that the system is inconsistent and has no solution.

CHECKPOINT Now try Exercise 29.

### Graphical Solution

Solve each equation for $y$ and use a graphing utility to graph the equations in the same viewing window.



The lines have the same slope but different $y$-intercepts, so they are parallel.

**Figure 7.11**

From Figure 7.11, you can conclude that the system has no solution. Note that when you use the *intersect* feature to find a point of intersection, the graphing utility cannot find a point of intersection and you will get an error message.

## Example 5 The Method of Elimination: Infinitely Many Solutions Case

Solve the system of linear equations.

$$\begin{cases} 2x - y = 1 & \text{Equation 1} \\ 4x - 2y = 2 & \text{Equation 2} \end{cases}$$

### Solution

To obtain coefficients that differ only in sign, multiply Equation 1 by $-2$.

$$\begin{array}{rcrl} 2x - y = 1 & \Rightarrow & -4x + 2y = -2 & \text{Multiply Equation 1 by } -2. \\ \underline{4x - 2y = 2} & \Rightarrow & \underline{4x - 2y = 2} & \text{Write Equation 2.} \\ & & 0 = 0 & \text{Add equations.} \end{array}$$

Because $0 = 0$ for all values of $x$ and $y$, the two equations turn out to be equivalent (have the same solution set). You can conclude that the system has infinitely many solutions. The solution set consists of all points $(x, y)$ lying on the line

$$2x - y = 1$$

as shown in Figure 7.12.

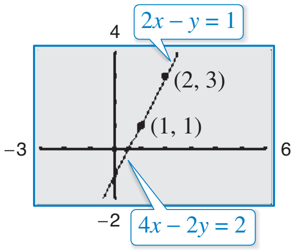


**Figure 7.12**

CHECKPOINT Now try Exercise 31.

In Example 4, note that the occurrence of the false statement $0 = 7$ indicates that the system has no solution. In Example 5, note that the occurrence of a statement that is true for all values of the variables—in this case, $0 = 0$—indicates that the system has infinitely many solutions.

## Application

At this point, you may be asking the question "How can I tell which application problems can be solved using a system of linear equations?" The answer comes from the following considerations.

1. Does the problem involve more than one unknown quantity?
2. Are there two (or more) equations or conditions to be satisfied?

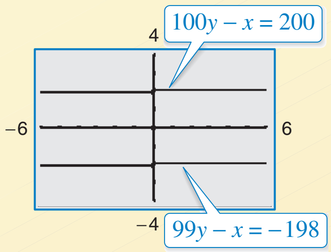
When one or both of these conditions are met, the appropriate mathematical model for the problem may be a system of linear equations.

### What's Wrong?

You use a graphing utility to graph the system

$$\begin{cases} 100y - x = 200 \\ 99y - x = -198 \end{cases}$$

as shown in the figure. You use the graph to conclude that the system has no solution. What's wrong?



### Example 6 Aviation

An airplane flying into a headwind travels the 2000-mile flying distance between Cleveland, Ohio and Fresno, California in 4 hours and 24 minutes. On the return flight, the same distance is traveled in 4 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

#### Solution

The two unknown quantities are the speeds of the wind and the plane. If $r_1$ is the speed of the plane and $r_2$ is the speed of the wind, then

$r_1 - r_2 =$ speed of the plane *against* the wind

$r_1 + r_2 =$ speed of the plane *with* the wind

as shown in Figure 7.13. Using the formula

distance = (rate)(time)

for these two speeds, you obtain the following equations.

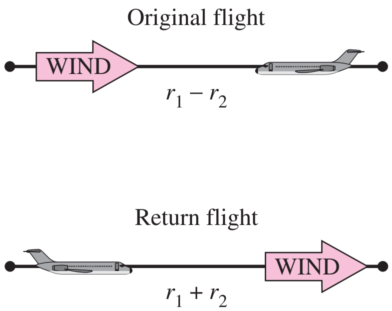


**Figure 7.13**

$$2000 = (r_1 - r_2)\left(4 + \frac{24}{60}\right)$$

$$2000 = (r_1 + r_2)(4)$$

These two equations simplify as follows.

$$\begin{cases} 5000 = 11r_1 - 11r_2 & \quad \text{Equation 1} \\ \phantom{0}500 = \phantom{11}r_1 + \phantom{11}r_2 & \quad \text{Equation 2} \end{cases}$$

To solve this system by elimination, multiply Equation 2 by 11.

$$\begin{array}{rcrl} 5000 = 11r_1 - 11r_2 & \Rightarrow & 5000 = 11r_1 - 11r_2 & \quad \text{Write Equation 1.} \\ \underline{500 = r_1 + r_2} & \Rightarrow & \underline{5500 = 11r_1 + 11r_2} & \quad \text{Multiply Equation 2 by 11.} \\ & & 10{,}500 = 22r_1 & \quad \text{Add equations.} \end{array}$$

So,

$$r_1 = \frac{10{,}500}{22} = \frac{5250}{11} \approx 477.27 \text{ miles per hour} \qquad \text{Speed of plane}$$

and

$$r_2 = 500 - \frac{5250}{11} = \frac{250}{11} \approx 22.73 \text{ miles per hour.} \qquad \text{Speed of wind}$$

Check this solution in the original statement of the problem.

✓CHECKPOINT Now try Exercise 77.

## 7.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1 and 2, fill in the blank(s).**

**1.** The first step in solving a system of equations by the ________ of ________ is to obtain coefficients for $x$ (or $y$) that differ only in sign.

**2.** Two systems of equations that have the same solution set are called ________ systems.

**3.** Is a system of linear equations with no solution consistent or inconsistent?

**4.** Is a system of linear equations with at least one solution consistent or inconsistent?

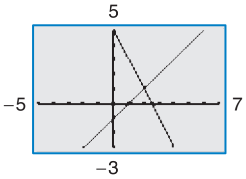
**5.** Is a system of two linear equations consistent when the lines are coincident?

**6.** When a system of linear equations has no solution, do the lines intersect?

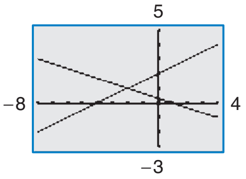
### Procedures and Problem Solving

**Solving a System by Elimination** **In Exercises 7–12, solve the system by the method of elimination. Label each line with its equation.**

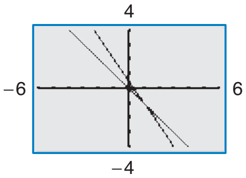
**7.** $\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$



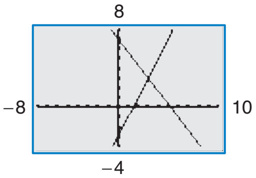
**8.** $\begin{cases} x + 3y = 1 \\ -x + 2y = 4 \end{cases}$



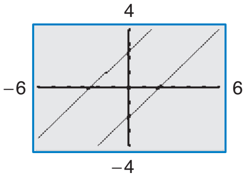
**9.** $\begin{cases} x + y = 0 \\ 3x + 2y = 1 \end{cases}$



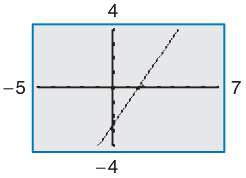
**10.** $\begin{cases} 2x - y = 3 \\ 4x + 3y = 21 \end{cases}$



**11.** $\begin{cases} x - y = 2 \\ -2x + 2y = 5 \end{cases}$



**12.** $\begin{cases} 3x - 2y = 5 \\ -6x + 4y = -10 \end{cases}$



**Solving a System by Elimination** **In Exercises 13–22, solve the system by the method of elimination and check any solutions algebraically.**

✓ **13.** $\begin{cases} x + 2y = 3 \\ x - 2y = 1 \end{cases}$

**14.** $\begin{cases} 3x - 5y = 2 \\ 2x + 5y = 13 \end{cases}$

✓ **15.** $\begin{cases} 2x + 3y = 18 \\ 5x - y = 11 \end{cases}$

**16.** $\begin{cases} x + 7y = 12 \\ 3x - 5y = 10 \end{cases}$

**17.** $\begin{cases} 3r + 2s = 10 \\ 2r + 5s = 3 \end{cases}$

**18.** $\begin{cases} 8r + 16s = 20 \\ 16r + 50s = 55 \end{cases}$

**19.** $\begin{cases} 5u + 6v = 24 \\ 3u + 5v = 18 \end{cases}$

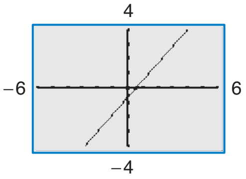
**20.** $\begin{cases} 3u + 11v = 4 \\ -2u - 5v = 9 \end{cases}$

**21.** $\begin{cases} 1.8x + 1.2y = 4 \\ 9x + 6y = 3 \end{cases}$

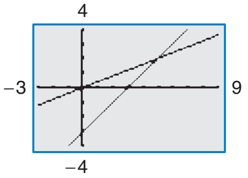
**22.** $\begin{cases} 3.1x - 2.9y = -10.2 \\ 31x - 12y = 34 \end{cases}$

**Recognizing Graphs of Linear Systems** **In Exercises 23–26, match the system of linear equations with its graph. State the number of solutions. Then state whether the system is consistent or inconsistent. [The graphs are labeled (a), (b), (c), and (d).]**

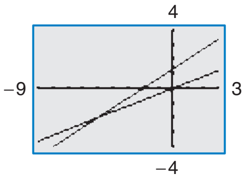
**(a)**



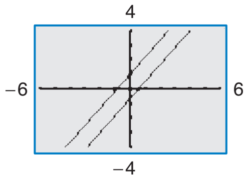
**(b)**



**(c)**



**(d)**



✓ **23.** $\begin{cases} 2x - 5y = 0 \\ x - y = 3 \end{cases}$

✓ **24.** $\begin{cases} -7x + 6y = -4 \\ 14x - 12y = 8 \end{cases}$

✓ **25.** $\begin{cases} 2x - 5y = 0 \\ 2x - 3y = -4 \end{cases}$

✓ **26.** $\begin{cases} 7x - 6y = -6 \\ -7x + 6y = -4 \end{cases}$

**Solving a System by Elimination** **In Exercises 27–46, solve the system by the method of elimination and check any solutions using a graphing utility.**

**27.** $\begin{cases} 5x + 3y = 6 \\ 3x - y = 5 \end{cases}$

**28.** $\begin{cases} x + 5y = 10 \\ 3x - 10y = -5 \end{cases}$

✓ **29.** $\begin{cases} \frac{2}{5}x - \frac{3}{2}y = 4 \\ \frac{1}{5}x - \frac{3}{4}y = -2 \end{cases}$

**30.** $\begin{cases} \frac{2}{3}x + \frac{1}{6}y = \frac{2}{3} \\ 4x + y = 4 \end{cases}$

✓ **31.** $\begin{cases} \frac{3}{4}x + y = \frac{1}{8} \\ \frac{9}{4}x + 3y = \frac{3}{8} \end{cases}$

**32.** $\begin{cases} \frac{1}{4}x + \frac{1}{6}y = 1 \\ -3x - 2y = 0 \end{cases}$

**33.** $\begin{cases} \frac{x+3}{4} + \frac{y-1}{3} = 1 \\ 2x - y = 12 \end{cases}$

**34.** $\begin{cases} \frac{x+2}{4} + \frac{y-1}{4} = 1 \\ x - y = 4 \end{cases}$

**35.** $\begin{cases} -5x + 6y = -3 \\ 20x - 24y = 12 \end{cases}$

**36.** $\begin{cases} 7x + 8y = 6 \\ -14x - 16y = -12 \end{cases}$

**37.** $\begin{cases} 2.5x - 3y = 1.5 \\ 2x - 2.4y = 1.2 \end{cases}$

**38.** $\begin{cases} 6.3x + 7.2y = 5.4 \\ 5.6x + 6.4y = 4.8 \end{cases}$

**39.** $\begin{cases} 0.2x - 0.5y = -27.8 \\ 0.3x + 0.4y = 68.7 \end{cases}$

**40.** $\begin{cases} 0.2x + 0.6y = -1 \\ x - 0.5y = 2 \end{cases}$

**41.** $\begin{cases} 0.05x - 0.03y = 0.21 \\ 0.07x + 0.02y = 0.16 \end{cases}$

**42.** $\begin{cases} 0.2x + 0.4y = -0.2 \\ x + 0.5y = -2.5 \end{cases}$

**43.** $\begin{cases} \frac{1}{x} + \frac{3}{y} = 2 \\ \frac{4}{x} - \frac{1}{y} = -5 \end{cases}$

**44.** $\begin{cases} \frac{2}{x} - \frac{1}{y} = 0 \\ \frac{4}{x} - \frac{3}{y} = -1 \end{cases}$

**45.** $\begin{cases} \frac{1}{x} + \frac{2}{y} = 5 \\ \frac{3}{x} - \frac{4}{y} = -5 \end{cases}$

**46.** $\begin{cases} \frac{2}{x} - \frac{1}{y} = 5 \\ \frac{6}{x} + \frac{1}{y} = 11 \end{cases}$

**Solving a System Graphically** **In Exercises 47–52, use a graphing utility to graph the lines in the system. Use the graphs to determine whether the system is consistent or inconsistent. If the system is consistent, determine the solution. Verify your results algebraically.**

**47.** $\begin{cases} 2x - 5y = 0 \\ x - y = 3 \end{cases}$

**48.** $\begin{cases} 2x + y = 5 \\ x - 2y = -1 \end{cases}$

**49.** $\begin{cases} \frac{3}{5}x - y = 3 \\ -3x + 5y = 9 \end{cases}$

**50.** $\begin{cases} 4x - 6y = 9 \\ \frac{16}{3}x - 8y = 12 \end{cases}$

**51.** $\begin{cases} 8x - 14y = 5 \\ 2x - 3.5y = 1.25 \end{cases}$

**52.** $\begin{cases} -x + 7y = 3 \\ -\frac{1}{7}x + y = 5 \end{cases}$

**Solving a System Graphically** **In Exercises 53–60, use a graphing utility to graph the two equations. Use the graphs to approximate the solution of the system. Round your results to three decimal places.**

**53.** $\begin{cases} 6y = 42 \\ 6x - y = 16 \end{cases}$

**54.** $\begin{cases} 4y = -8 \\ 7x - 2y = 25 \end{cases}$

**55.** $\begin{cases} \frac{3}{2}x - \frac{1}{5}y = 8 \\ -2x + 3y = 3 \end{cases}$

**56.** $\begin{cases} \frac{3}{4}x - \frac{5}{2}y = -9 \\ -x + 6y = 28 \end{cases}$

**57.** $\begin{cases} \frac{1}{3}x + y = -\frac{1}{3} \\ 5x - 3y = 7 \end{cases}$

**58.** $\begin{cases} 5x - y = -4 \\ 2x + \frac{3}{5}y = \frac{2}{5} \end{cases}$

**59.** $\begin{cases} 0.5x + 2.2y = 9 \\ 6x + 0.4y = -22 \end{cases}$

**60.** $\begin{cases} 2.4x + 3.8y = -17.6 \\ 4x - 0.2y = -3.2 \end{cases}$

**Solving a System** **In Exercises 61–68, use any method to solve the system.**

**61.** $\begin{cases} 3x - 5y = 7 \\ 2x + y = 9 \end{cases}$

**62.** $\begin{cases} -x + 3y = 17 \\ 4x + 3y = 7 \end{cases}$

**63.** $\begin{cases} y = 2x - 5 \\ y = 5x - 11 \end{cases}$

**64.** $\begin{cases} 7x + 3y = 16 \\ y = x + 2 \end{cases}$

**65.** $\begin{cases} x - 5y = 21 \\ 6x + 5y = 21 \end{cases}$

**66.** $\begin{cases} y = -2x - 17 \\ y = 2 - 3x \end{cases}$

**67.** $\begin{cases} -5x + 9y = 13 \\ y = x - 4 \end{cases}$

**68.** $\begin{cases} 4x - 3y = 6 \\ -5x + 7y = -1 \end{cases}$

**Exploration** **In Exercises 69–72, find a system of linear equations that has the given solution. (There are many correct answers.)**

**69.** $(0, 8)$

**70.** $(3, -4)$

**71.** $\left(3, \frac{5}{2}\right)$

**72.** $\left(-\frac{2}{3}, -10\right)$

**Economics** **In Exercises 73–76, find the *point of equilibrium* of the demand and supply equations. The point of equilibrium is the price $p$ and the number of units $x$ that satisfy both the demand and supply equations.**

| | *Demand* | *Supply* |
|---|---|---|
| **73.** | $p = 500 - 0.4x$ | $p = 380 + 0.1x$ |
| **74.** | $p = 100 - 0.05x$ | $p = 25 + 0.1x$ |
| **75.** | $p = 140 - 0.00002x$ | $p = 80 + 0.00001x$ |
| **76.** | $p = 400 - 0.0002x$ | $p = 225 + 0.0005x$ |

✓ **77.** **Aviation** An airplane flying into a headwind travels the 1800-mile flying distance between New York City and Albuquerque, New Mexico in 3 hours and 36 minutes. On the return flight, the same distance is traveled in 3 hours. Find the airspeed of the plane and the speed of the wind, assuming that both remain constant.

**78.** **Nutrition** Two cheeseburgers and one small order of French fries from a fast-food restaurant contain a total of 830 calories. Three cheeseburgers and two small orders of French fries contain a total of 1360 calories. Find the number of calories in each item.

**79.** **Business** A minor league baseball team had a total attendance one evening of 1175. The tickets for adults and children sold for \$5.00 and \$3.50, respectively. The ticket revenue that night was \$5087.50.

(a) Create a system of linear equations to find the numbers of adults $A$ and children $C$ at the game.

(b) Solve your system of equations by elimination or by substitution. Explain your choice.

(c) Use the *intersect* feature or the *zoom* and *trace* features of a graphing utility to solve your system.

**80.** **Chemistry** Thirty liters of a 40% acid solution is obtained by mixing a 25% solution with a 50% solution.

(a) Write a system of equations in which one equation represents the amount of final mixture required and the other represents the percent of acid in the final mixture. Let $x$ and $y$ represent the amounts of the 25% and 50% solutions, respectively.

(b) Use a graphing utility to graph the two equations in part (a) in the same viewing window. As the amount of the 25% solution increases, how does the amount of the 50% solution change?

(c) How much of each solution is required to obtain the specified concentration of the final mixture?

**81.** **Business** A grocer sells oranges for \$0.95 each and grapefruits for \$1.05 each. You purchased a mix of 16 oranges and grapefruits and paid \$15.90. How many of each type of fruit did you buy?

**82.** **Sales** The sales $S$ (in millions of dollars) of Fossil and Aeropostale clothing stores from 2000 to 2008 can be modeled by

$$\begin{cases} S - 138.98t = 413.5 \\ S - 212.35t = 135.3 \end{cases}$$

where $t$ is the year, with $t = 0$ corresponding to 2000. (Sources: Fossil, Inc.; Aeropostale, Inc.)

(a) Solve the system of equations using the method of your choice. Explain why you chose that method.

(b) Interpret the meaning of the solution in the context of the problem.

**83.** **Economics** Revenues for a movie rental store on a particular Friday evening are \$867.50 for 310 rentals. The rental fee for movies is \$3.00 each and the rental fee for video games is \$2.50 each. Determine the number of each type that are rented that evening.

**84.** ***Why you should learn it*** *(p.480)* On a Saturday night, the manager of a shoe store evaluates the receipts of the previous week's sales. Two hundred fifty pairs of two different styles of running shoes were sold. One style sold for \$75.50 and the other sold for \$89.95. The receipts totaled \$20,031. The cash register that was supposed to record the number of each type of shoe sold malfunctioned. Can you recover the information? If so, how many shoes of each type were sold?

**Fitting a Line to Data** **To find the least squares regression line $y = ax + b$ for a set of points**

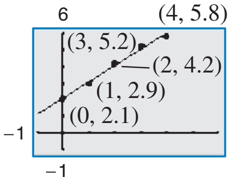
$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

**you can solve the following system for $a$ and $b$.**

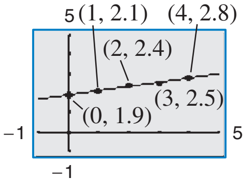
$$\begin{cases} nb + \left(\sum_{i=1}^{n} x_i\right)a = \left(\sum_{i=1}^{n} y_i\right) \\ \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \left(\sum_{i=1}^{n} x_i y_i\right) \end{cases}$$

**In Exercises 85–88, the sums have been evaluated. Solve the given system for $a$ and $b$ to find the least squares regression line for the points. Use a graphing utility to confirm the result.**

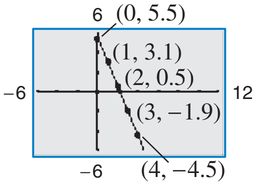
**85.** $\begin{cases} 5b + 10a = 20.2 \\ 10b + 30a = 50.1 \end{cases}$



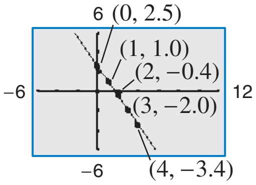
**86.** $\begin{cases} 5b + 10a = 11.7 \\ 10b + 30a = 25.6 \end{cases}$



**87.** $\begin{cases} 5b + 10a = 2.7 \\ 10b + 30a = -19.6 \end{cases}$



**88.** $\begin{cases} 5b + 10a = -2.3 \\ 10b + 30a = -19.4 \end{cases}$



**89.** **MODELING DATA**

Four test plots were used to explore the relationship between wheat yield (in bushels per acre) and the amount of fertilizer applied (in hundreds of pounds per acre). The results are shown in the table.

| Fertilizer, $x$ | Yield, $y$ |
|---|---|
| 1.0 | 32 |
| 1.5 | 41 |
| 2.0 | 48 |
| 2.5 | 53 |

(a) Find the least squares regression line $y = ax + b$ for the data by solving the system for $a$ and $b$.

$$\begin{cases} 4b + 7.0a = 174 \\ 7b + 13.5a = 322 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to confirm the result in part (a).

(c) Use the graphing utility to plot the data and graph the linear model from part (a) in the same viewing window.

(d) Use the linear model from part (a) to predict the yield for a fertilizer application of 160 pounds per acre.

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**90. MODELING DATA**

A candy store manager wants to know the demand for a candy bar as a function of the price. The daily sales for different prices of the product are shown in the table.

| Price, $x$ | Demand, $y$ |
|---|---|
| \$1.00 | 45 |
| \$1.20 | 37 |
| \$1.50 | 23 |

(a) Find the least squares regression line $y = ax + b$ for the data by solving the system for $a$ and $b$.

$$\begin{cases} 3.00b + 3.70a = 105.00 \\ 3.70b + 4.69a = 123.90 \end{cases}$$

(b) Use the *regression* feature of a graphing utility to confirm the result in part (a).

(c) Use the graphing utility to plot the data and graph the linear model from part (a) in the same viewing window.

(d) Use the linear model from part (a) to predict the demand when the price is \$1.75.

## Conclusions

**True or False?** **In Exercises 91–93, determine whether the statement is true or false. Justify your answer.**

**91.** If a system of linear equations has two distinct solutions, then it has an infinite number of solutions.

**92.** If a system of linear equations has no solution, then the lines must be parallel.

**93.** Solving a system of equations graphically using a graphing utility always yields an exact solution.

**94.** **Writing** Briefly explain whether or not it is possible for a consistent system of linear equations to have exactly two solutions.

**95.** **Think About It** Give examples of (a) a system of linear equations that has no solution and (b) a system of linear equations that has an infinite number of solutions. (There are many correct answers.)

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**96.** **CAPSTONE** Find all value(s) of $k$ for which the system of linear equations

$$\begin{cases} x + 3y = 9 \\ 2x + 6y = k \end{cases}$$

has (a) infinitely many solutions and (b) no solution.

**Solving a System** **In Exercises 97 and 98, solve the system of equations for $u$ and $v$. While solving for these variables, consider the transcendental functions as constants. (Systems of this type are found in a course in differential equations.)**

**97.** $\begin{cases} u \sin x + v \cos x = 0 \\ u \cos x - v \sin x = \sec x \end{cases}$

**98.** $\begin{cases} u \cos 2x + v \sin 2x = 0 \\ u(-2 \sin 2x) + v(2 \cos 2x) = \csc 2x \end{cases}$

## Cumulative Mixed Review

**Solving an Inequality** **In Exercises 99–104, solve the inequality and graph the solution on a real number line.**

**99.** $-11 - 6x \geq 33$

**100.** $-6 \leq 3x - 10 < 6$

**101.** $|x - 8| < 10$

**102.** $|x + 10| \geq -3$

**103.** $2x^2 + 3x - 35 < 0$

**104.** $3x^2 + 12x > 0$

**Rewriting a Logarithmic Expression** **In Exercises 105–110, write the expression as the logarithm of a single quantity.**

**105.** $\ln x + \ln 6$

**106.** $\ln x - 5 \ln(x + 3)$

**107.** $\log_9 12 - \log_9 x$

**108.** $\frac{1}{4} \log_6 3 + \frac{1}{4} \log_6 x$

**109.** $2 \ln x - \ln(x + 2)$

**110.** $\frac{1}{2} \ln(x^2 + 4) - \ln x$

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**111.** ***Make a Decision*** To work an extended application analyzing the average undergraduate tuition, room, and board charges at private colleges in the United States from 1985 through 2008, visit this textbook's *Companion Website*. (Data Source: National Center for Education Statistics)

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# 7.3 Multivariable Linear Systems

## Row-Echelon Form and Back-Substitution

The method of elimination can be applied to a system of linear equations in more than two variables. When elimination is used to solve a system of linear equations, the goal is to rewrite the system in a form to which back-substitution can be applied. To see how this works, consider the following two systems of linear equations.

**System of Three Linear Equations in Three Variables** (See Example 2):

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

**Equivalent System in Row-Echelon Form** (See Example 1):

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$$

The second system is said to be in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. After comparing the two systems, it should be clear that it is easier to solve the system in row-echelon form, using back-substitution.

### What you should learn

- Use back-substitution to solve linear systems in row-echelon form.
- Use Gaussian elimination to solve systems of linear equations.
- Solve nonsquare systems of linear equations.
- Graphically interpret three-variable linear systems.
- Use systems of linear equations to write partial fraction decompositions of rational expressions.
- Use systems of linear equations in three or more variables to model and solve real-life problems

### Why you should learn it

Systems of linear equations in three or more variables can be used to model and solve real-life problems. For instance, Exercise 90 on page 501 shows how to use a system of linear equations to analyze the numbers of par-3, par-4, and par-5 holes on a golf course.

### Example 1 Using Back-Substitution in Row-Echelon Form

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ y + 4z = 7 & \text{Equation 2} \\ z = 2 & \text{Equation 3} \end{cases}$$

#### Solution

From Equation 3, you know the value of $z$. To solve for $y$, substitute $z = 2$ into Equation 2 to obtain

$$y + 4(2) = 7 \qquad \text{Substitute 2 for } z.$$

$$y = -1. \qquad \text{Solve for } y.$$

Next, substitute $y = -1$ and $z = 2$ into Equation 1 to obtain

$$x - 2(-1) + 3(2) = 9 \qquad \text{Substitute } -1 \text{ for } y \text{ and 2 for } z.$$

$$x = 1. \qquad \text{Solve for } x.$$

The solution is

$$x = 1, \quad y = -1, \quad \text{and} \quad z = 2$$

which can be written as the **ordered triple**

$$(1, -1, 2).$$

Check this in the original system of equations.

CHECKPOINT Now try Exercise 13.

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## Gaussian Elimination

Two systems of equations are *equivalent* when they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that *is* in row-echelon form by using one or more of the elementary row operations shown below. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

### Elementary Row Operations for Systems of Equations

**1.** Interchange two equations.

**2.** Multiply one of the equations by a nonzero constant.

**3.** Add a multiple of one equation to another equation.

### Example 2 Using Gaussian Elimination to Solve a System

Solve the system of linear equations.

$$\begin{cases} x - 2y + 3z = 9 & \text{Equation 1} \\ -x + 3y + z = -2 & \text{Equation 2} \\ 2x - 5y + 5z = 17 & \text{Equation 3} \end{cases}$$

### Solution

Because the leading coefficient of the first equation is 1, you can begin by saving the $x$ at the upper left and eliminating the other $x$-terms from the first column.

**Study Tip**

Arithmetic errors are often made when elementary row operations are performed. You should note the operation performed in each step so that you can go back and check your work.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 2x - 5y + 5z = 17 \end{cases}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ -y - z = -1 \end{cases}$$

Adding $-2$ times the first equation to the third equation produces a new third equation.

Now that all but the first $x$ have been eliminated from the first column, go to work on the second column. (You need to eliminate $y$ from the third equation.)

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 3z = 6 \end{cases}$$

Adding the second equation to the third equation produces a new third equation.

Finally, you need a coefficient of 1 for $z$ in the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$$

Multiplying the third equation by $\frac{1}{3}$ produces a new third equation.

This is the same system that was solved in Example 1. As in that example, you can conclude that the solution is

$$x = 1, \quad y = -1, \quad \text{and} \quad z = 2$$

written as $(1, -1, 2)$.

✓CHECKPOINT Now try Exercise 21.

The goal of Gaussian elimination is to use elementary row operations on a system in order to isolate one variable. You can then solve for the value of the variable and use back-substitution to find the values of the remaining variables.

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage in the elimination process, you obtain a false statement such as

$$0 = -2. \qquad \text{False statement}$$

## Example 3 An Inconsistent System

Solve the system of linear equations.

$$\begin{cases} x - 3y + z = 1 & \text{Equation 1} \\ 2x - y - 2z = 2 & \text{Equation 2} \\ x + 2y - 3z = -1 & \text{Equation 3} \end{cases}$$

### Solution

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ x + 2y - 3z = -1 \end{cases}$$

Adding −2 times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 5y - 4z = -2 \end{cases}$$

Adding −1 times the first equation to the third equation produces a new third equation.

$$\begin{cases} x - 3y + z = 1 \\ 5y - 4z = 0 \\ 0 = -2 \end{cases}$$

Adding −1 times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is a false statement, you can conclude that this system is inconsistent and so has no solution. Moreover, because this system is equivalent to the original system, you can conclude that the original system also has no solution.

✓CHECKPOINT Now try Exercise 27.

As with a system of linear equations in two variables, the number of solutions of a system of linear equations in more than two variables must fall into one of three categories.

### The Number of Solutions of a Linear System

For a system of linear equations, exactly one of the following is true.

1. There is exactly one solution.
2. There are infinitely many solutions.
3. There is no solution.

A system of linear equations is called *consistent* when it has at least one solution. A consistent system with exactly one solution is **independent.** A consistent system with infinitely many solutions is **dependent.** A system of linear equations is called *inconsistent* when it has no solution.

## Example 4 A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{cases} x + y - 3z = -1 & \text{Equation 1} \\ \phantom{x + {}} y - z = 0 & \text{Equation 2} \\ -x + 2y \phantom{{}-z} = 1 & \text{Equation 3} \end{cases}$$

### Solution

$$\begin{cases} x + y - 3z = -1 \\ \phantom{x + {}} y - z = 0 \\ \phantom{x + {}} 3y - 3z = 0 \end{cases}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{cases} x + y - 3z = -1 \\ \phantom{x + {}} y - z = 0 \\ \phantom{x + y - 3z} 0 = 0 \end{cases}$$

Adding $-3$ times the second equation to the third equation produces a new third equation.

This result means that Equation 3 depends on Equations 1 and 2 in the sense that it gives no additional information about the variables. So, the original system is equivalent to

$$\begin{cases} x + y - 3z = -1 \\ \phantom{x + {}} y - z = 0 \end{cases}.$$

In the last equation, solve for $y$ in terms of $z$ to obtain $y = z$. Back-substituting for $y$ in the previous equation produces

$$x = 2z - 1.$$

Finally, letting $z = a$, where $a$ is a real number, the solutions of the original system are all of the form

$$x = 2a - 1, \quad y = a, \quad \text{and} \quad z = a.$$

So, every ordered triple of the form

$$(2a - 1, a, a) \qquad a \text{ is a real number.}$$

is a solution of the system.

**CHECKPOINT** Now try Exercise 31.

**Study Tip**

There are an infinite number of solutions to Example 4, but they are all of a specific form. By selecting, for instance, $a$-values of 0, 1, and 3, you can verify that $(-1, 0, 0)$, $(1, 1, 1)$, and $(5, 3, 3)$ are specific solutions. It is incorrect to say simply that the solution to Example 4 is "infinite." You must also specify the form of the solutions.

In Example 4, there are other ways to write the same infinite set of solutions. For instance, the solutions could have been written as

$$\left(b, \tfrac{1}{2}(b + 1), \tfrac{1}{2}(b + 1)\right). \qquad b \text{ is a real number.}$$

This description produces the same set of solutions, as shown below.

| *Substitution* | *Solution* | |
|---|---|---|
| $a = 0$ | $(2(0) - 1, 0, 0) = (-1, 0, 0)$ | Same solution |
| $b = -1$ | $\left(-1, \tfrac{1}{2}(-1 + 1), \tfrac{1}{2}(-1 + 1)\right) = (-1, 0, 0)$ | |
| $a = 1$ | $(2(1) - 1, 1, 1) = (1, 1, 1)$ | Same solution |
| $b = 1$ | $\left(1, \tfrac{1}{2}(1 + 1), \tfrac{1}{2}(1 + 1)\right) = (1, 1, 1)$ | |
| $a = 2$ | $(2(2) - 1, 2, 2) = (3, 2, 2)$ | Same solution |
| $b = 3$ | $\left(3, \tfrac{1}{2}(3 + 1), \tfrac{1}{2}(3 + 1)\right) = (3, 2, 2)$ | |
| $a = 3$ | $(2(3) - 1, 3, 3) = (5, 3, 3)$ | Same solution |
| $b = 5$ | $\left(5, \tfrac{1}{2}(5 + 1), \tfrac{1}{2}(5 + 1)\right) = (5, 3, 3)$ | |

## Nonsquare Systems

So far, each system of linear equations you have looked at has been *square*, which means that the number of equations is equal to the number of variables. In a **nonsquare system of equations,** the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

### Example 5 A System with Fewer Equations than Variables

Solve the system of linear equations.

$$\begin{cases} x - 2y + z = 2 & \text{Equation 1} \\ 2x - y - z = 1 & \text{Equation 2} \end{cases}$$

### Solution

Begin by rewriting the system in row-echelon form.

$$\begin{cases} x - 2y + z = 2 \\ 3y - 3z = -3 \end{cases}$$

Adding $-2$ times the first equation to the second equation produces a new second equation.

$$\begin{cases} x - 2y + z = 2 \\ y - z = -1 \end{cases}$$

Multiplying the second equation by $\frac{1}{3}$ produces a new second equation.

Solve for $y$ in terms of $z$ to obtain

$$y = z - 1.$$

By back-substituting into Equation 1, you can solve for $x$ as follows.

$$\begin{aligned} x - 2y + z &= 2 && \text{Equation 1} \\ x - 2(z - 1) + z &= 2 && \text{Substitute for } y. \\ x - 2z + 2 + z &= 2 && \text{Distributive Property} \\ x &= z && \text{Solve for } x. \end{aligned}$$

Finally, by letting $z = a$ where $a$ is a real number, you have the solution

$$x = a, \quad y = a - 1, \quad \text{and} \quad z = a.$$

So, every ordered triple of the form

$$(a, a - 1, a) \qquad a \text{ is a real number.}$$

is a solution of the system.

**CHECKPOINT** Now try Exercise 39.

In Example 5, try choosing some values of $a$ to obtain different solutions of the system, such as $(1, 0, 1)$, $(2, 1, 2)$, and $(3, 2, 3)$. Then check each of the solutions in the original system, as follows.

*Check:* $(1, 0, 1)$

$$\begin{aligned} 1 - 2(0) + 1 &\stackrel{?}{=} 2 \\ 2 &= 2 \ ✓ \\ 2(1) - 0 - 1 &\stackrel{?}{=} 1 \\ 1 &= 1 \ ✓ \end{aligned}$$

*Check:* $(2, 1, 2)$

$$\begin{aligned} 2 - 2(1) + 2 &\stackrel{?}{=} 2 \\ 2 &= 2 \ ✓ \\ 2(2) - 1 - 2 &\stackrel{?}{=} 1 \\ 1 &= 1 \ ✓ \end{aligned}$$

*Check:* $(3, 2, 3)$

$$\begin{aligned} 3 - 2(2) + 3 &\stackrel{?}{=} 2 \\ 2 &= 2 \ ✓ \\ 2(3) - 2 - 3 &\stackrel{?}{=} 1 \\ 1 &= 1 \ ✓ \end{aligned}$$

## Graphical Interpretation of Three-Variable Systems

Solutions of equations in three variables can be represented graphically using a **three-dimensional coordinate system.** To construct such a system, begin with the $xy$-coordinate plane in a horizontal position. Then draw the $z$-axis as a vertical line through the origin.

Every ordered triple $(x, y, z)$ corresponds to a point on the three-dimensional coordinate system. For instance, the points corresponding to $(-2, 5, 4)$, $(2, -5, 3)$, and $(3, 3, -2)$ are shown in Figure 7.14.

The **graph of an equation in three variables** consists of all points $(x, y, z)$ that are solutions of the equation. The graph of a linear equation in three variables is a *plane*. Sketching graphs on a three-dimensional coordinate system is difficult because the sketch itself is only two-dimensional.

One technique for sketching a plane is to find the three points at which the plane intersects the axes. For instance, the plane

$$3x + 2y + 4z = 12$$

intersects the $x$-axis at the point $(4, 0, 0)$, the $y$-axis at the point $(0, 6, 0)$, and the $z$-axis at the point $(0, 0, 3)$. By plotting these three points, connecting them with line segments, and shading the resulting triangular region, you can sketch a portion of the graph, as shown in Figure 7.15.

The graph of a system of three linear equations in three variables consists of *three* planes. When these planes intersect in a single point, the system has exactly one solution (see Figure 7.16). When the three planes have no point in common, the system has no solution (see Figures 7.17 and 7.18). When the three planes intersect in a line or a plane, the system has infinitely many solutions (see Figures 7.19 and 7.20).

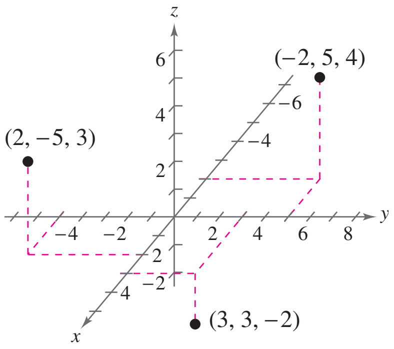


**Figure 7.14**

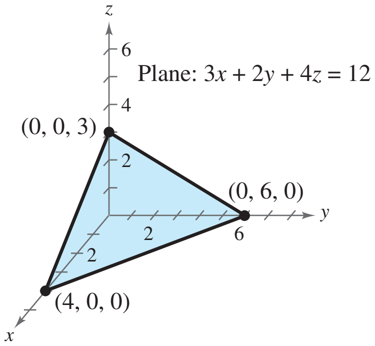
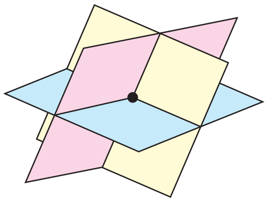
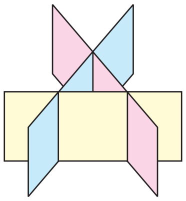


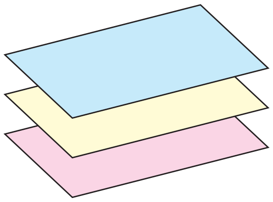
**Figure 7.15**



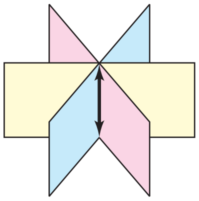
***Solution: One point***
**Figure 7.16**



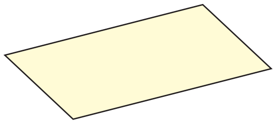
***Solution: None***
**Figure 7.17**



***Solution: None***
**Figure 7.18**



***Solution: One line***
**Figure 7.19**



***Solution: One plane***
**Figure 7.20**

### Technology Tip

Three-dimensional graphing utilities and computer algebra systems, such as *Maple* and *Mathematica*, are very efficient in producing three-dimensional graphs. They are good tools to use while studying calculus. If you have access to such a utility, try reproducing the plane shown in Figure 7.15.

## Partial Fraction Decomposition

A rational expression can often be written as the sum of two or more simpler rational expressions. For example, the rational expression

$$\frac{x+7}{x^2-x-6}$$

can be written as the sum of two fractions with linear denominators. That is,

$$\frac{x+7}{x^2-x-6} = \underbrace{\frac{2}{x-3}}_{\text{Partial fraction}} + \underbrace{\frac{-1}{x+2}}_{\text{Partial fraction}}.$$

Each fraction on the right side of the equation is a **partial fraction,** and together they make up the **partial fraction decomposition** of the left side.

### Decomposition of $N(x)/D(x)$ into Partial Fractions

**1.** *Divide if improper:* If

$$\frac{N(x)}{D(x)}$$

is an improper fraction [degree of $N(x) \geq$ degree of $D(x)$], then divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 (below) to the proper rational expression

$$\frac{N_1(x)}{D(x)}.$$

**2.** *Factor denominator:* Completely factor the denominator into factors of the form

$$(px+q)^m \qquad \text{and} \qquad (ax^2+bx+c)^n$$

where $(ax^2+bx+c)$ is irreducible over the reals.

**3.** *Linear factors:* For *each* factor of the form

$$(px+q)^m$$

the partial fraction decomposition must include the following sum of $m$ fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdot \cdot \cdot + \frac{A_m}{(px+q)^m}$$

**4.** *Quadratic factors:* For *each* factor of the form

$$(ax^2+bx+c)^n$$

the partial fraction decomposition must include the following sum of $n$ fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdot \cdot \cdot + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

One of the most important applications of partial fractions is in calculus. Partial fractions can be used in a calculus operation called antidifferentiation.

### Example 6 Partial Fraction Decomposition: Distinct Linear Factors

Write the partial fraction decomposition of

$$\frac{x+7}{x^2-x-6}.$$

### Solution

The expression is proper, so factor the denominator. Because

$$x^2 - x - 6 = (x-3)(x+2)$$

you should include one partial fraction with a constant numerator for each linear factor of the denominator and write

$$\frac{x+7}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}.$$

Multiplying each side of this equation by the least common denominator

$$(x-3)(x+2)$$

leads to the basic equation

$$\begin{aligned} x + 7 &= A(x+2) + B(x-3) && \text{Basic equation} \\ &= Ax + 2A + Bx - 3B && \text{Distributive Property} \\ &= (A+B)x + 2A - 3B. && \text{Write in polynomial form.} \end{aligned}$$

Because two polynomials are equal if and only if the coefficients of like terms are equal, you can equate the coefficients of like terms to opposite sides of the equation.

$$x + 7 = (A+B)x + (2A - 3B) \qquad \text{Equate coefficients of like terms.}$$

You can now write the following system of linear equations.

$$\begin{cases} A + B = 1 & \text{Equation 1} \\ 2A - 3B = 7 & \text{Equation 2} \end{cases}$$

You can solve the system of linear equations as follows.

$$\begin{array}{lcll} A + B = 1 & \Rightarrow & 3A + 3B = 3 & \text{Multiply Equation 1 by 3.} \\ \underline{2A - 3B = 7} & \Rightarrow & \underline{2A - 3B = 7} & \text{Write Equation 2.} \\ & & 5A \phantom{- 3B} = 10 & \text{Add equations.} \end{array}$$

From this equation, you can see that

$$A = 2.$$

By back-substituting this value of $A$ into Equation 1, you can solve for $B$ as follows.

$$\begin{aligned} A + B &= 1 && \text{Write Equation 1.} \\ 2 + B &= 1 && \text{Substitute 2 for} \\ B &= -1 && \text{Solve for } B. \end{aligned}$$

So, the partial fraction decomposition is

$$\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}.$$

Check this result by combining the two partial fractions on the right side of the equation, or by using a graphing utility.

**CHECKPOINT** Now try Exercise 61.

### Technology Tip

You can graphically check the decomposition found in Example 6. To do this, use a graphing utility to graph

$$y_1 = \frac{x+7}{x^2-x-6} \quad \text{and}$$

$$y_2 = \frac{2}{x-3} - \frac{1}{x+2}$$

in the same viewing window. The graphs should be identical.

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The next example shows how to find the partial fraction decomposition for a rational function whose denominator has a repeated linear factor.

### Example 7 Partial Fraction Decomposition: Repeated Linear Factors

Write the partial fraction decomposition of

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

#### Solution

The expression is proper, so factor the denominator. Because the denominator factors as

$$\begin{aligned} x^3 + 2x^2 + x &= x(x^2 + 2x + 1) \\ &= x(x + 1)^2 \end{aligned}$$

you should include one partial fraction with a constant numerator for each power of $x$ and $(x + 1)$ and write

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}.$$

Multiplying by the LCD

$$x(x + 1)^2$$

leads to the basic equation

$$\begin{aligned} 5x^2 + 20x + 6 &= A(x + 1)^2 + Bx(x + 1) + Cx && \text{Basic equation} \\ &= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx && \text{Expand.} \\ &= (A + B)x^2 + (2A + B + C)x + A. && \text{Polynomial form} \end{aligned}$$

Equating coefficients of like terms on opposite sides of the equation

$$5x^2 + 20x + 6 = (A + B)x^2 + (2A + B + C)x + A$$

produces the following system of linear equations.

$$\begin{cases} A + B \quad\quad\;\, = 5 & \text{Equation 1} \\ 2A + B + C = 20 & \text{Equation 2} \\ A \quad\quad\quad\quad\;\, = 6 & \text{Equation 3} \end{cases}$$

Substituting 6 for $A$ in Equation 1 yields

$$\begin{aligned} 6 + B &= 5 \\ B &= -1. \end{aligned}$$

Substituting 6 for $A$ and $-1$ for $B$ in Equation 2 yields

$$\begin{aligned} 2(6) + (-1) + C &= 20 \\ C &= 9. \end{aligned}$$

So, the partial fraction decomposition is

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{6}{x} - \frac{1}{x + 1} + \frac{9}{(x + 1)^2}.$$

Check this result by combining the three partial fractions on the right side of the equation, or by using a graphing utility.

**CHECKPOINT** Now try Exercise 65.

### Explore the Concept

Partial fraction decomposition is practical only for rational functions whose denominators factor "nicely." For example, the factorization of the expression $x^2 - x - 5$ is

$$\left(x - \frac{1 - \sqrt{21}}{2}\right)\left(x - \frac{1 + \sqrt{21}}{2}\right).$$

Write the basic equation and try to complete the decomposition for

$$\frac{x + 7}{x^2 - x - 5}.$$

What problems do you encounter?

## Applications

### Example 8 Vertical Motion

The height at time $t$ of an object that is moving in a (vertical) line with constant acceleration $a$ is given by the *position equation*

$$s = \tfrac{1}{2}at^2 + v_0t + s_0.$$

The height $s$ is measured in feet, the acceleration $a$ is measured in feet per second squared, $t$ is measured in seconds, $v_0$ is the initial velocity (in feet per second) at $t = 0$, and $s_0$ is the initial height (in feet). Find the values of $a$, $v_0$, and $s_0$ when

$$s = 52 \text{ at } t = 1, \quad s = 52 \text{ at } t = 2, \quad \text{and } s = 20 \text{ at } t = 3$$

and interpret the result. (See Figure 7.21.)

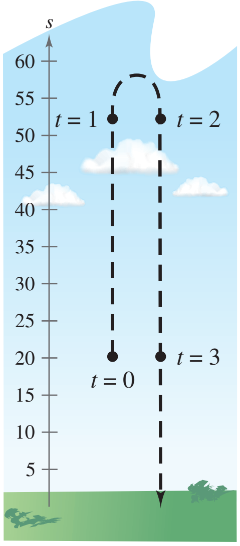


Figure 7.21

#### Solution

You can obtain three linear equations in $a$, $v_0$, and $s_0$ as follows.

$$\begin{aligned}
&\text{When } t = 1\text{: } \tfrac{1}{2}a(1)^2 + v_0(1) + s_0 = 52 &&\Rightarrow &a + 2v_0 + 2s_0 &= 104\\
&\text{When } t = 2\text{: } \tfrac{1}{2}a(2)^2 + v_0(2) + s_0 = 52 &&\Rightarrow &2a + 2v_0 + s_0 &= 52\\
&\text{When } t = 3\text{: } \tfrac{1}{2}a(3)^2 + v_0(3) + s_0 = 20 &&\Rightarrow &9a + 6v_0 + 2s_0 &= 40
\end{aligned}$$

Solving this system yields $a = -32$, $v_0 = 48$, and $s_0 = 20$. This solution results in a position equation of

$$s = -16t^2 + 48t + 20$$

and implies that the object was thrown upward at a velocity of 48 feet per second from a height of 20 feet.

✓CHECKPOINT Now try Exercise 73.

### Example 9 Data Analysis: Curve-Fitting

Find a quadratic equation

$$y = ax^2 + bx + c$$

whose graph passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$.

#### Solution

Because the graph of $y = ax^2 + bx + c$ passes through the points $(-1, 3)$, $(1, 1)$, and $(2, 6)$, you can write the following.

$$\begin{aligned}
&\text{When } x = -1,\ y = 3\text{:} && a(-1)^2 + b(-1) + c = 3\\
&\text{When } x = 1,\ \ \ y = 1\text{:} && a(1)^2 + b(1) + c = 1\\
&\text{When } x = 2,\ \ \ y = 6\text{:} && a(2)^2 + b(2) + c = 6
\end{aligned}$$

This produces the following system of linear equations.

$$\begin{cases}
a - b + c = 3 & \text{Equation 1}\\
a + b + c = 1 & \text{Equation 2}\\
4a + 2b + c = 6 & \text{Equation 3}
\end{cases}$$

The solution of this system is $a = 2$, $b = -1$, and $c = 0$. So, the equation of the parabola is $y = 2x^2 - x$, and its graph is shown in Figure 7.22.

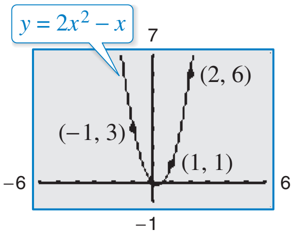


Figure 7.22

✓CHECKPOINT Now try Exercise 77.

## 7.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1–6, fill in the blank.**

1. A system of equations that is in _______ form has a "stair-step" pattern with leading coefficients of 1.
2. A solution of a system of three linear equations in three unknowns can be written as an _______ , which has the form $(x, y, z)$.
3. The process used to write a system of equations in row-echelon form is called _______ elimination.
4. A system of equations is called _______ when the number of equations differs from the number of variables in the system.
5. Solutions of equations in three variables can be pictured using a _______ coordinate system.
6. The process of writing a rational expression as the sum of two or more simpler rational expressions is called _______ .

7. Is a consistent system with exactly one solution *independent* or *dependent*?
8. Is a consistent system with infinitely many solutions *independent* or *dependent*?

### Procedures and Problem Solving

**Checking Solutions** **In Exercises 9–12, determine whether each ordered triple is a solution of the system of equations.**

**9.** $$\begin{cases} 3x - y + z = 1 \\ 2x - 3z = -14 \\ 5y + 2z = 8 \end{cases}$$

(a) $(3, 5, -3)$ (b) $(-1, 0, 4)$

(c) $(0, -1, 3)$ (d) $(1, 0, 4)$

**10.** $$\begin{cases} 3x + 4y - z = 17 \\ 5x - y + 2z = -2 \\ 2x - 3y + 7z = -21 \end{cases}$$

(a) $(1, 5, 6)$ (b) $(-2, -4, 2)$

(c) $(1, 3, -2)$ (d) $(0, 7, 0)$

**11.** $$\begin{cases} 4x + y - z = 0 \\ -8x - 6y + z = -\frac{7}{4} \\ 3x - y = -\frac{9}{4} \end{cases}$$

(a) $(0, 1, 1)$ (b) $\left(-\frac{3}{2}, \frac{5}{4}, -\frac{5}{4}\right)$

(c) $\left(-\frac{1}{2}, \frac{3}{4}, -\frac{5}{4}\right)$ (d) $\left(-\frac{1}{2}, \frac{1}{6}, -\frac{3}{4}\right)$

**12.** $$\begin{cases} -4x - y - 8z = -6 \\ y + z = 0 \\ 4x - 7y = 6 \end{cases}$$

(a) $(-2, -2, 2)$ (b) $\left(-\frac{33}{2}, -10, 10\right)$

(c) $\left(\frac{1}{8}, -\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(-\frac{11}{2}, -4, 4\right)$

**Using Back-Substitution** **In Exercises 13–18, use back-substitution to solve the system of linear equations.**

✓ **13.** $$\begin{cases} 2x - y + 5z = 16 \\ y + 2z = 2 \\ z = 2 \end{cases}$$

**14.** $$\begin{cases} 4x - 3y - 2z = 21 \\ 6y - 5z = -8 \\ z = -2 \end{cases}$$

**15.** $$\begin{cases} 2x + y - 3z = 10 \\ y + z = 12 \\ z = 2 \end{cases}$$

**16.** $$\begin{cases} x - y + 2z = 22 \\ 3y - 8z = -9 \\ z = -3 \end{cases}$$

**17.** $$\begin{cases} 4x - 2y + z = 8 \\ -y + z = 4 \\ z = 11 \end{cases}$$

**18.** $$\begin{cases} 5x - 8z = 22 \\ 3y - 5z = 10 \\ z = -4 \end{cases}$$

**Performing Row Operations** **In Exercises 19 and 20, perform the row operation and write the equivalent system. What did the operation accomplish?**

**19.** Add Equation 1 to Equation 2.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

**20.** Add $-2$ times Equation 1 to Equation 3.

$$\begin{cases} x - 2y + 3z = 5 & \text{Equation 1} \\ -x + 3y - 5z = 4 & \text{Equation 2} \\ 2x - 3z = 0 & \text{Equation 3} \end{cases}$$

**Solving a System of Linear Equations** In Exercises 21–42, solve the system of linear equations and check any solution algebraically.

✓ **21.** $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x - z = 0 \end{cases}$

**22.** $\begin{cases} x + y + z = 3 \\ x - 2y + 4z = 5 \\ 3y + 4z = 5 \end{cases}$

**23.** $\begin{cases} 2x + 2z = 2 \\ 5x + 3y = 4 \\ 3y - 4z = 4 \end{cases}$

**24.** $\begin{cases} 2x + 4y + z = 1 \\ x - 2y - 3z = 2 \\ x + y - z = -1 \end{cases}$

**25.** $\begin{cases} 4x + y - 3z = 11 \\ 2x - 3y + 2z = 9 \\ x + y + z = -3 \end{cases}$

**26.** $\begin{cases} 2x + 4y + z = -4 \\ 2x - 4y + 6z = 13 \\ 4x - 2y + z = 6 \end{cases}$

✓ **27.** $\begin{cases} 3x - 2y + 4z = 1 \\ x + y - 2z = 3 \\ 2x - 3y + 6z = 8 \end{cases}$

**28.** $\begin{cases} 5x - 3y + 2z = 3 \\ 2x + 4y - z = 7 \\ x - 11y + 4z = 3 \end{cases}$

**29.** $\begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 0 \\ 5x + 9y + 17z = 0 \end{cases}$

**30.** $\begin{cases} 2x + y + 3z = 1 \\ 2x + 6y + 8z = 3 \\ 6x + 8y + 18z = 5 \end{cases}$

✓ **31.** $\begin{cases} 3x - 3y + 6z = 6 \\ x + 2y - z = 5 \\ 5x - 8y + 13z = 7 \end{cases}$

**32.** $\begin{cases} x + 4z = 13 \\ 4x - 2y + z = 7 \\ 2x - 2y - 7z = -19 \end{cases}$

**33.** $\begin{cases} x - 2y + 3z = 4 \\ 3x - y + 2z = 0 \\ x + 3y - 4z = -2 \end{cases}$

**34.** $\begin{cases} -x + 3y + z = 4 \\ 4x - 2y - 5z = -7 \\ 2x + 4y - 3z = 12 \end{cases}$

**35.** $\begin{cases} x + 4z = 1 \\ x + y + 10z = 10 \\ 2x - y + 2z = -5 \end{cases}$

**36.** $\begin{cases} 3x - 2y - 6z = -4 \\ -3x + 2y + 6z = 1 \\ x - y - 5z = -3 \end{cases}$

**37.** $\begin{cases} x + 2y + z = 1 \\ x - 2y + 3z = -3 \\ 2x + y + z = -1 \end{cases}$

**38.** $\begin{cases} x - 2y + z = 2 \\ 2x + 2y - 3z = -4 \\ 5x + z = 1 \end{cases}$

✓ **39.** $\begin{cases} x - 2y + 5z = 2 \\ 4x - z = 0 \end{cases}$

**40.** $\begin{cases} 12x + 5y + z = 0 \\ 23x + 4y - z = 0 \end{cases}$

**41.** $\begin{cases} 2x - 3y + z = -2 \\ -4x + 9y = 7 \end{cases}$

**42.** $\begin{cases} 10x - 3y + 2z = 0 \\ 19x - 5y - z = 0 \end{cases}$

**Exploration** In Exercises 43–46, find a system of linear equations that has the given solution. (There are many correct answers.)

**43.** $(3, -4, 2)$

**44.** $(-5, -2, 1)$

**45.** $\left(-6, -\frac{1}{2}, -\frac{7}{4}\right)$

**46.** $\left(-\frac{3}{2}, 4, -7\right)$

**Sketching a Plane** In Exercises 47–50, sketch the plane represented by the linear equation. Then list four points that lie in the plane.

**47.** $2x + 3y + 4z = 12$

**48.** $x + y + z = 6$

**49.** $2x + y + z = 4$

**50.** $x + 2y + 2z = 6$

**Writing the Partial Fraction Decomposition** In Exercises 51–56, write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

**51.** $\dfrac{7}{x^2 - 14x}$

**52.** $\dfrac{x - 2}{x^2 + 4x + 3}$

**53.** $\dfrac{12}{x^3 - 10x^2}$

**54.** $\dfrac{x^2 - 3x + 2}{4x^3 + 11x^2}$

**55.** $\dfrac{4x^2 + 3}{(x - 5)^3}$

**56.** $\dfrac{6x + 5}{(x + 2)^4}$

**Partial Fraction Decomposition** In Exercises 57–70, write the partial fraction decomposition for the rational expression. Check your result algebraically by combining fractions, and check your result graphically by using a graphing utility to graph the rational expression and the partial fractions in the same viewing window.

**57.** $\dfrac{1}{x^2 - 1}$

**58.** $\dfrac{1}{4x^2 - 9}$

**59.** $\dfrac{1}{x^2 + x}$

**60.** $\dfrac{3}{x^2 - 3x}$

✓ **61.** $\dfrac{5 - x}{2x^2 + x - 1}$

**62.** $\dfrac{x - 2}{x^2 + 4x + 3}$

**63.** $\dfrac{x^2 + 12x + 12}{x^3 - 4x}$

**64.** $\dfrac{x^2 + 12x - 9}{x^3 - 9x}$

✓ **65.** $\dfrac{4x^2 + 2x - 1}{x^2(x + 1)}$

**66.** $\dfrac{2x - 3}{(x - 1)^2}$

**67.** $\dfrac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2}$

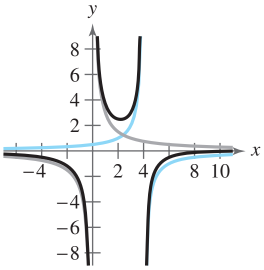
**68.** $\dfrac{x^3 + 2x^2 - x + 1}{x^2 + 3x - 4}$

**69.** $\dfrac{x^4}{(x - 1)^3}$

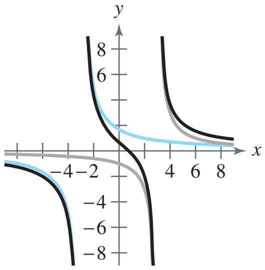
**70.** $\dfrac{4x^4}{(2x - 1)^3}$

**Writing the Partial Fraction Decomposition** In Exercises 71 and 72, write the partial fraction decomposition for the rational function. Identify the graph of the rational function and the graph of each term of its decomposition. State any relationship between the vertical asymptotes of the rational function and the vertical asymptotes of the terms of the decomposition.

**71.** $y = \dfrac{x - 12}{x(x - 4)}$



**72.** $y = \dfrac{2(4x - 3)}{x^2 - 9}$



**Vertical Motion** **In Exercises 73–76, an object moving vertically is at the given heights at the specified times. Find the position equation $s = \frac{1}{2}at^2 + v_0 t + s_0$ for the object.**

✓ **73.** At $t = 1$ second, $s = 128$ feet.
At $t = 2$ seconds, $s = 80$ feet.
At $t = 3$ seconds, $s = 0$ feet.

**74.** At $t = 1$ second, $s = 32$ feet.
At $t = 2$ seconds, $s = 32$ feet.
At $t = 3$ seconds, $s = 0$ feet.

**75.** At $t = 1$ second, $s = 352$ feet.
At $t = 2$ seconds, $s = 272$ feet.
At $t = 3$ seconds, $s = 160$ feet.

**76.** At $t = 1$ second, $s = 132$ feet.
At $t = 2$ seconds, $s = 100$ feet.
At $t = 3$ seconds, $s = 36$ feet.

**Data Analysis: Curve-Fitting** **In Exercises 77–80, find the equation of the parabola**

$$y = ax^2 + bx + c$$

**that passes through the points. To verify your result, use a graphing utility to plot the points and graph the parabola.**

✓ **77.** $(0, 0), (2, -2), (4, 0)$ **78.** $(0, 3), (1, 4), (2, 3)$

**79.** $(2, 0), (3, -1), (4, 0)$

**80.** $(-2, -3), (-1, 0), \left(\frac{1}{2}, -3\right)$

**Finding the Equation of a Circle** **In Exercises 81–84, find the equation of the circle**

$$x^2 + y^2 + Dx + Ey + F = 0$$

**that passes through the points. To verify your result, use a graphing utility to plot the points and graph the circle.**

**81.** $(0, 0), (5, 5), (10, 0)$ **82.** $(0, 0), (0, 6), (3, 3)$

**83.** $(-3, -1), (2, 4), (-6, 8)$

**84.** $(0, 0), (0, -2), (3, 0)$

**85.** **Finance** A small corporation borrowed \$775,000 to expand its software line. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate given that the annual interest was \$67,500 and the amount borrowed at 8% was four times the amount borrowed at 10%?

**86.** **Finance** A small corporation borrowed \$800,000 to expand its line of toys. Some of the money was borrowed at 8%, some at 9%, and some at 10%. How much was borrowed at each rate given that the annual interest was \$67,000 and the amount borrowed at 8% was five times the amount borrowed at 10%?

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**Investment Portfolio** **In Exercises 87 and 88, consider an investor with a portfolio totaling \$500,000 that is invested in certificates of deposit, municipal bonds, blue-chip stocks, and growth or speculative stocks. How much is invested in each type of investment?**

**87.** The certificates of deposit pay 3% annually, and the municipal bonds pay 5% annually. Over a five-year period, the investor expects the blue-chip stocks to return 8% annually and the growth stocks to return 10% annually. The investor wants a combined annual return of 5% and also wants to have only one-fourth of the portfolio invested in stocks.

**88.** The certificates of deposit pay 2% annually, and the municipal bonds pay 4% annually. Over a five-year period, the investor expects the blue-chip stocks to return 10% annually and the growth stocks to return 14% annually. The investor wants a combined annual return of 6% and also wants to have only one-fourth of the portfolio invested in stocks.

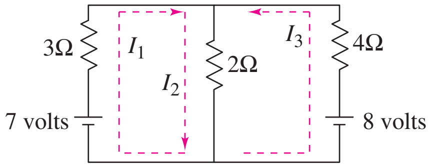
**89.** **Physical Education** In the 2010 Women's NCAA Championship basketball game, the University of Connecticut defeated Stanford University by a score of 53 to 47. Connecticut won by scoring a combination of two-point baskets, three-point baskets, and one-point free throws. The number of two-point baskets was four more than the number of free throws. The number of free throws was three more than the number of three-point baskets. What combination of scoring accounted for Connecticut's 53 points? (Source: NCAA)

**90.** ***Why you should learn it*** *(p. 489)* The Augusta National Golf Club in Augusta, Georgia is an 18-hole course that consists of par-3 holes, par-4 holes, and par-5 holes. A golfer who shoots par has a total of 72 strokes for the entire course. There are two more par-4 holes than twice the number of par-5 holes, and the number of par-3 holes is equal to the number of par-5 holes. Find the numbers of par-3, par-4, and par-5 holes on the course. (Source: Augusta National, Inc.)

**91.** **Electrical Engineering** When Kirchhoff's Laws are applied to the electrical network in the figure, the currents $I_1$, $I_2$, and $I_3$ are the solution of the system

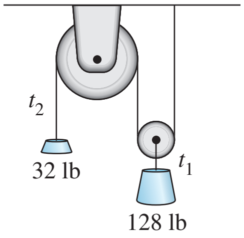
$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 \qquad = 7. \\ \qquad 2I_2 + 4I_3 = 8 \end{cases}$$

Find the currents.



**92. Physics** A system of pulleys is loaded with 128-pound and 32-pound weights (see figure). The tensions $t_1$ and $t_2$ in the ropes and the acceleration $a$ of the 32-pound weight are modeled by the following system, where $t_1$ and $t_2$ are measured in pounds and $a$ is in feet per second squared. Solve the system.

$$\begin{cases} t_1 - 2t_2 \qquad = 0 \\ t_1 \qquad - 2a = 128 \\ \qquad t_2 + a = 32 \end{cases}$$



**Fitting a Parabola** **To find the least squares regression parabola $y = ax^2 + bx + c$ for a set of points**

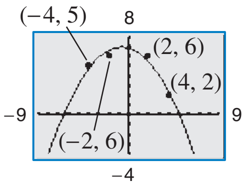
$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

**you can solve the following system of linear equations for $a$, $b$, and $c$.**

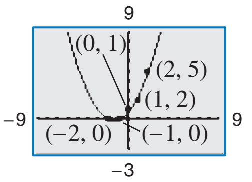
$$\begin{cases} nc + \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)c + \left(\sum_{i=1}^{n} x_i^2\right)b + \left(\sum_{i=1}^{n} x_i^3\right)a = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i^2\right)c + \left(\sum_{i=1}^{n} x_i^3\right)b + \left(\sum_{i=1}^{n} x_i^4\right)a = \sum_{i=1}^{n} x_i^2 y_i \end{cases}$$

**In Exercises 93–96, the sums have been evaluated. Solve the given system for $a$ and $b$ to find the least squares regression parabola for the points. Use a graphing utility to confirm the result.**

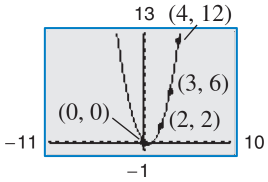
**93.** $\begin{cases} 4c \qquad + 40a = 19 \\ \quad 40b \qquad = -12 \\ 40c \qquad + 544a = 160 \end{cases}$



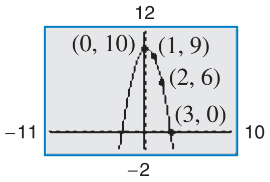
**94.** $\begin{cases} 5c \qquad + 10a = 8 \\ \quad 10b \qquad = 12 \\ 10c \qquad + 34a = 22 \end{cases}$



**95.** $\begin{cases} 4c + 9b + 29a = 20 \\ 9c + 29b + 99a = 70 \\ 29c + 99b + 353a = 254 \end{cases}$



**96.** $\begin{cases} 4c + 6b + 14a = 25 \\ 6c + 14b + 36a = 21 \\ 14c + 36b + 98a = 33 \end{cases}$



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**97. MODELING DATA**

During the testing of a new automobile braking system, the speeds $x$ (in miles per hour) and the stopping distances $y$ (in feet) were recorded in the table.

| **Speed, $x$** | **Stopping distance, $y$** |
|---|---|
| 30 | 55 |
| 40 | 105 |
| 50 | 188 |

(a) Use the data to create a system of linear equations. Then find the least squares regression parabola for the data by solving the system.

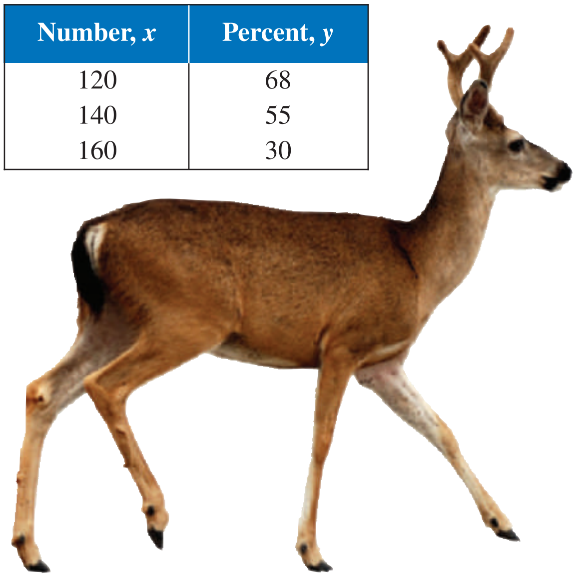
(b) Use a graphing utility to graph the parabola and the data in the same viewing window.

(c) Use the model to estimate the stopping distance for a speed of 70 miles per hour.

**98. MODELING DATA**

A wildlife management team studied the reproduction rates of deer in three five-acre tracts of a wildlife preserve. In each tract, the number of females $x$ and the percent of females $y$ that had offspring the following year were recorded. The results are shown in the table.

| **Number, $x$** | **Percent, $y$** |
|---|---|
| 120 | 68 |
| 140 | 55 |
| 160 | 30 |



(a) Use the data to create a system of linear equations. Then find the least squares regression parabola for the data by solving the system.

(b) Use a graphing utility to graph the parabola and the data in the same viewing window.

(c) Use the model to predict the percent of females that had offspring when there were 170 females.

**99. Thermodynamics** The magnitude of the range $R$ of exhaust temperatures (in degrees Fahrenheit) in an experimental diesel engine is approximated by the model

$$R = \frac{2000(4 - 3x)}{(11 - 7x)(7 - 4x)}, \quad 0 \le x \le 1$$

where $x$ is the relative load (in foot-pounds).

(a) Write the partial fraction decomposition of the rational function.

(b) The decomposition in part (a) is the difference of two fractions. The absolute values of the terms give the expected maximum and minimum temperatures of the exhaust gases. Use a graphing utility to graph each term.

**100. Environment** The predicted cost $C$ (in thousands of dollars) for a company to remove $p\%$ of a chemical from its waste water is given by the model

$$C = \frac{120p}{10{,}000 - p^2}, \quad 0 \le p < 100.$$

Write the partial fraction decomposition of the rational function. Verify your result by using the *table* feature of a graphing utility to create a table comparing the original function with the partial fractions.

## Conclusions

**True or False? In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.**

**101.** The system

$$\begin{cases} x + 4y - 5z = 8 \\ \phantom{x + }2y + \phantom{5}z = 5 \\ \phantom{x + 4y + 5}z = 1 \end{cases}$$

is in row-echelon form.

**102.** If a system of three linear equations is inconsistent, then its graph has no points common to all three equations.

**103. Error Analysis** You are tutoring a student in algebra. In trying to find a partial fraction decomposition, your student writes the following.

$$\frac{x^2 + 1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}$$

$$x^2 + 1 = A(x - 1) + Bx \qquad \text{Basic equation}$$

$$x^2 + 1 = (A + B)x - A$$

Your student then forms the following system of linear equations.

$$\begin{cases} A + B = 0 \\ -A \phantom{+ B} = 1 \end{cases}$$

Solve the system and check the partial fraction decomposition it yields. Has your student worked the problem correctly? If not, what went wrong?

**104. CAPSTONE** Find values of $a$, $b$, and $c$ (if possible) such that the system of linear equations has (a) a unique solution, (b) no solution, and (c) an infinite number of solutions.

$$\begin{cases} x + y \phantom{+ cz} = 2 \\ \phantom{x + } y + z = 2 \\ x \phantom{+ y} + z = 2 \\ ax + by + cz = 0 \end{cases}$$

**105. Think About It** Are the two systems of equations equivalent? Give reasons for your answer.

$$\begin{cases} x + 3y - z = 6 \\ 2x - y + 2z = 1 \\ 3x + 2y - z = 2 \end{cases} \qquad \begin{cases} x + 3y - z = 6 \\ -7y + 4z = 1 \\ -7y - 4z = -16 \end{cases}$$

**106. Writing** When using Gaussian elimination to solve a system of linear equations, explain how you can recognize that the system has no solution. Give an example that illustrates your answer.

**Lagrange Multiplier In Exercises 107 and 108, find values of $x$, $y$, and $\lambda$ that satisfy the system. These systems arise in certain optimization problems in calculus. ($\lambda$ is called a *Lagrange multiplier*.)**

**107.** $\begin{cases} y + \lambda = 0 \\ x + \lambda = 0 \\ x + y - 10 = 0 \end{cases}$

**108.** $\begin{cases} 2x + \lambda = 0 \\ 2y + \lambda = 0 \\ x + y - 4 = 0 \end{cases}$

## Cumulative Mixed Review

**Finding Zeros In Exercises 109–112, (a) determine the real zeros of $f$ and (b) sketch the graph of $f$.**

**109.** $f(x) = x^3 + x^2 - 12x$

**110.** $f(x) = -8x^4 + 32x^2$

**111.** $f(x) = 2x^3 + 5x^2 - 21x - 36$

**112.** $f(x) = 6x^3 - 29x^2 - 6x + 5$

**Solving a Trigonometric Equation In Exercises 113 and 114, solve the equation.**

**113.** $4\sqrt{3}\tan\theta - 3 = 1$

**114.** $6\cos x - 2 = 1$

**115. *Make a Decision*** To work an extended application analyzing the earnings per share for Wal-Mart Stores, Inc. from 1988 through 2008, visit this textbook's *Companion Website*. (Data Source: Wal-Mart Stores, Inc.)

# 7.4 Matrices and Systems of Equations

## Matrices

In this section, you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of real numbers called a **matrix.** The plural of matrix is *matrices.*

### Definition of Matrix

If $m$ and $n$ are positive integers, then an $m \times n$ (read "$m$ by $n$") matrix is a rectangular array

$$\begin{array}{cc} & \begin{array}{ccccc}\text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n\end{array} \\ \begin{array}{c}\text{Row 1}\\ \text{Row 2}\\ \text{Row 3}\\ \vdots \\ \text{Row } m\end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}\end{array}$$

in which each **entry** $a_{ij}$ of the matrix is a real number. An $m \times n$ matrix has $m$ rows and $n$ columns.

The entry in the $i$th row and $j$th column is denoted by the *double subscript* notation $a_{ij}$. For instance, the entry $a_{23}$ is the entry in the second row and third column. A matrix having $m$ rows and $n$ columns is said to be of **dimension** $m \times n$. If $m = n$, then the matrix is **square** of dimension $m \times m$ (or $n \times n$). For a square matrix, the entries

$$a_{11}, a_{22}, a_{33}, \ . \ . \ .$$

are the **main diagonal** entries.

### Example 1 Dimension of a Matrix

Determine the dimension of each matrix.

**a.** $[2]$ **b.** $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

**d.** $\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$ **e.** $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

#### Solution

**a.** This matrix has *one* row and *one* column. The dimension of the matrix is 1 × 1.

**b.** This matrix has *one* row and *four* columns. The dimension of the matrix is 1 × 4.

**c.** This matrix has *two* rows and *two* columns. The dimension of the matrix is 2 × 2.

**d.** This matrix has *three* rows and *two* columns. The dimension of the matrix is 3 × 2.

**e.** This matrix has *three* rows and *one* column. The dimension of the matrix is 3 × 1.

✓CHECKPOINT Now try Exercise 9.

A matrix that has only one row [such as the matrix in Example 1(b)] is called a **row matrix,** and a matrix that has only one column [such as the matrix in Example 1(e)] is called a **column matrix.**

### What you should learn

- Write matrices and identify their dimensions.
- Perform elementary row operations on matrices.
- Use matrices and Gaussian elimination to solve systems of linear equations.
- Use matrices and Gauss-Jordan elimination to solve systems of linear equations.

### Why you should learn it

Matrices can be used to solve systems of linear equations in two or more variables. For instance, Exercise 91 on page 516 shows how a matrix can be used to help model an equation for the average retail price of prescription drugs.

**Pharmacist**

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A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but not including the constant terms) is the **coefficient matrix** of the system. The matrix derived from the constant terms of the system is the **constant matrix** of the system.

*System:*

$$\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x - 4z = 6 \end{cases}$$

*Augmented Matrix:*

$$\left[\begin{array}{rrrcr} 1 & -4 & 3 & \vdots & 5 \\ -1 & 3 & -1 & \vdots & -3 \\ 2 & 0 & -4 & \vdots & 6 \end{array}\right]$$

*Coefficient Matrix:*

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

*Constant Matrix:*

$$\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}$$

Note the use of 0 for the missing coefficient of the $y$-variable in the third equation, and also note the fourth column (of constant terms) in the augmented matrix. The optional dotted line in the augmented matrix helps to separate the coefficients of the linear system from the constant terms.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using 0's for any missing coefficients of variables.

## Example 2 Writing an Augmented Matrix

Write the augmented matrix for the system of linear equations.

$$\begin{cases} x + 3y = 9 \\ -y + 4z = -2 \\ x - 5z = 0 \end{cases}$$

What is the dimension of the augmented matrix?

### Solution

Begin by writing the linear system and aligning the variables.

$$\begin{cases} x + 3y \phantom{{}+4z} = 9 \\ \phantom{x} - y + 4z = -2 \\ x \phantom{{}+3y} - 5z = 0 \end{cases}$$

Next, use the coefficients and constant terms as the matrix entries. Include zeros for the coefficients of the missing variables.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{rrrcr} 1 & 3 & 0 & \vdots & 9 \\ 0 & -1 & 4 & \vdots & -2 \\ 1 & 0 & -5 & \vdots & 0 \end{array}\right]$$

The augmented matrix has three rows and four columns, so it is a 3 × 4 matrix. The notation $R_n$ is used to designate each row in the matrix. For instance, Row 1 is represented by $R_1$.

CHECKPOINT Now try Exercise 15.

## Elementary Row Operations

In Section 7.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system.

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a sequence of elementary row operations.

### Technology Tip

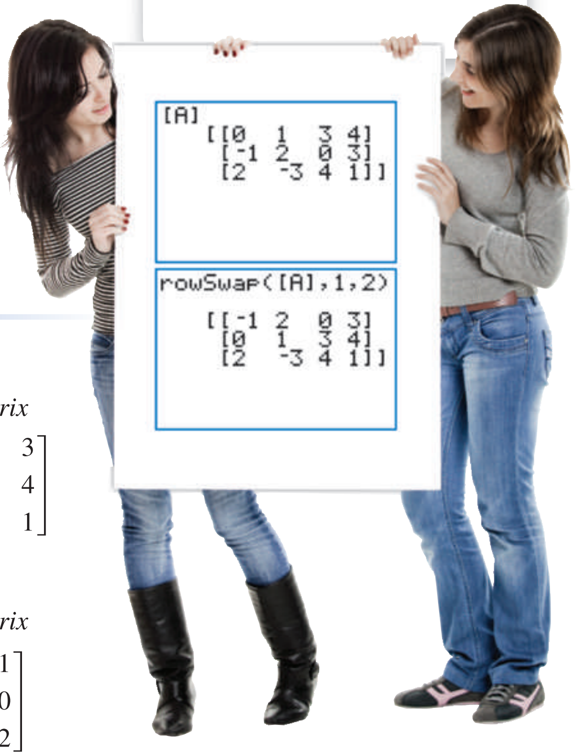
Most graphing utilities can perform elementary row operations on matrices. For instructions on how to use the *matrix* feature and the elementary row operations features of a graphing utility, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

### Elementary Row Operations for Matrices

1. Interchange two rows.
2. Multiply one of the rows by a nonzero constant.
3. Add a multiple of one row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting the elementary row operations performed in each step so that you can go back and check your work.

Example 3 demonstrates the elementary row operations described above.



```
[A]
   [[0  1  3 4]
    [-1 2  0 3]
    [2  -3 4 1]]
```

```
rowSwap([A],1,2)
   [[-1 2  0 3]
    [0  1  3 4]
    [2  -3 4 1]]
```

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## Example 3 Elementary Row Operations

**a.** Interchange the first and second rows of the original matrix.

*Original Matrix*

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

*New Row-Equivalent Matrix*

$$\begin{array}{c} R_2 \\ R_1 \\ \\ \end{array} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

**b.** Multiply the first row of the original matrix by $\frac{1}{2}$.

*Original Matrix*

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

*New Row-Equivalent Matrix*

$$\begin{array}{c} \frac{1}{2}R_1 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

**c.** Add $-2$ times the first row of the original matrix to the third row.

*Original Matrix*

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

*New Row-Equivalent Matrix*

$$\begin{array}{c} \\ \\ -2R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

Note that the elementary row operation is written beside the row that is *changed*.

✓CHECKPOINT Now try Exercise 25.

## Gaussian Elimination with Back-Substitution

In Example 2 of Section 7.3, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The basic difference between the two methods is that with matrices you do not need to keep writing the variables.

### Example 4 Comparing Linear Systems and Matrix Operations

*Linear System*

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

*Associated Augmented Matrix*

$$\left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 1 & \vdots & -2 \\ 2 & -5 & 5 & \vdots & 17 \end{array}\right]$$

Add the first equation to the second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 2x - 5y + 5z = 17 \end{cases}$$

Add the first row to the second row: $R_1 + R_2$.

$$\begin{array}{r} \\ R_1 + R_2 \rightarrow \\ \\ \end{array}\left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 4 & \vdots & 7 \\ 2 & -5 & 5 & \vdots & 17 \end{array}\right]$$

Add $-2$ times the first equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ -y - z = -1 \end{cases}$$

Add $-2$ times the first row to the third row: $-2R_1 + R_3$.

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \end{array}\left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 4 & \vdots & 7 \\ 0 & -1 & -1 & \vdots & -1 \end{array}\right]$$

Add the second equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ 3z = 6 \end{cases}$$

Add the second row to the third row: $R_2 + R_3$.

$$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \end{array}\left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 4 & \vdots & 7 \\ 0 & 0 & 3 & \vdots & 6 \end{array}\right]$$

Multiply the third equation by $\frac{1}{3}$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 4z = 7 \\ z = 2 \end{cases}$$

Multiply the third row by $\frac{1}{3}$: $\frac{1}{3}R_3$.

$$\begin{array}{r} \\ \\ \frac{1}{3}R_3 \rightarrow \end{array}\left[\begin{array}{rrrcr} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 4 & \vdots & 7 \\ 0 & 0 & 1 & \vdots & 2 \end{array}\right]$$

At this point, you can use back-substitution to find that the solution is

$$x = 1,\ y = -1, \text{ and } z = 2$$

as was done in Example 2 of Section 7.3.

**CHECKPOINT** Now try Exercise 31.

Remember that you should check a solution by substituting the values of $x$, $y$, and $z$ into each equation in the original system. For instance, you can check the solution to Example 4 as follows.

*Equation 1*

$$\begin{aligned} x - 2y + 3z &= 9 \\ 1 - 2(-1) + 3(2) &\overset{?}{=} 9 \\ 9 &= 9 \ ✓ \end{aligned}$$

*Equation 2*

$$\begin{aligned} -x + 3y + z &= -2 \\ -1 + 3(-1) + 2 &\overset{?}{=} -2 \\ -2 &= -2 \ ✓ \end{aligned}$$

*Equation 3*

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ 2(1) - 5(-1) + 5(2) &\overset{?}{=} 17 \\ 17 &= 17 \ ✓ \end{aligned}$$

The last matrix in Example 4 is in **row-echelon form.** The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in this form, a matrix must have the following properties.

### Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

It is worth mentioning that the row-echelon form of a matrix is not unique. That is, two different sequences of elementary row operations may yield different row-echelon forms. The *reduced* row-echelon form of a given matrix, however, is unique.

### Technology Tip

Some graphing utilities can automatically transform a matrix to row-echelon form and reduced row-echelon form. For instructions on how to use the *row-echelon form* feature and the *reduced row-echelon form* feature of a graphing utility, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

## Example 5 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

**a.** $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

**c.** $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**e.** $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

### Solution

The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because the row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not a leading 1.

✓CHECKPOINT Now try Exercise 35.

Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 5, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$. What elementary row operation could you perform on the matrix in part (b) so that it would be in row-echelon form?

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. You should operate *from left to right by columns*, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

## Example 6 Gaussian Elimination with Back-Substitution

Solve the system of equations.

$$\begin{cases} \phantom{x + 2}y + \phantom{4}z - 2w = -3 \\ x + 2y - \phantom{4}z \phantom{- 3w} = \phantom{-}2 \\ 2x + 4y + \phantom{4}z - 3w = -2 \\ x - 4y - 7z - \phantom{3}w = -19 \end{cases}$$

### Solution

$$\left[\begin{array}{rrrr:r} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

Write augmented matrix.

$$\begin{array}{r} \curvearrowleft R_2 \\ R_1 \\ \\ \\ \end{array} \left[\begin{array}{rrrr:r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{array}\right]$$

Interchange $R_1$ and $R_2$ so first column has leading 1 in upper left corner.

$$\begin{array}{r} \\ \\ -2R_1 + R_3 \rightarrow \\ -R_1 + R_4 \rightarrow \end{array} \left[\begin{array}{rrrr:r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{array}\right]$$

Perform operations on $R_3$ and $R_4$ so first column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \\ 6R_2 + R_4 \rightarrow \end{array} \left[\begin{array}{rrrr:r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{array}\right]$$

Perform operations on $R_4$ so second column has zeros below its leading 1.

$$\begin{array}{r} \\ \\ \frac{1}{3}R_3 \rightarrow \\ -\frac{1}{13}R_4 \rightarrow \end{array} \left[\begin{array}{rrrr:r} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

Perform operations on $R_3$ and $R_4$ so third and fourth columns have leading 1's.

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{cases} x + 2y - z \phantom{- 2w} = 2 \\ \phantom{x + 2}y + z - 2w = -3 \\ \phantom{x + 2y + }z - \phantom{2}w = -2 \\ \phantom{x + 2y + z - 2}w = 3 \end{cases}.$$

Using back-substitution, you can determine that the solution is

$$x = -1, \quad y = 2, \quad z = 1, \quad \text{and} \quad w = 3.$$

Check this in the original system of equations.

CHECKPOINT Now try Exercise 61.

The following steps summarize the procedure used in Example 6.

### Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution.

Remember that it is possible for a system to have no solution. If, in the elimination process, you obtain a row with zeros except for the last entry, then you can conclude that the system is inconsistent.

## Example 7 A System with No Solution

Solve the system of equations.

$$\begin{cases} x - y + 2z = 4 \\ x + z = 6 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 1 \end{cases}$$

### Solution

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{array}\right]$$

Write augmented matrix.

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \\ -3R_1 + R_4 \rightarrow \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{array}\right]$$

Perform row operations.

$$\begin{array}{r} \\ \\ R_2 + R_3 \rightarrow \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{array}\right]$$

Perform row operations.

Note that the third row of this matrix consists of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. You can see why this is true by converting back to a system of linear equations.

$$\begin{cases} x - y + 2z = 4 \\ y - z = 2 \\ 0 = -2 \\ 5y - 7z = -11 \end{cases}$$

Because the third equation

$$0 = -2$$

is not possible, the system has no solution.

**CHECKPOINT** Now try Exercise 59.

## Gauss–Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form of the matrix. A second method of elimination, called **Gauss-Jordan elimination** after Carl Friedrich Gauss (1777–1855) and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in Example 8.

### Example 8 Gauss–Jordan Elimination

Use Gauss-Jordan elimination to solve the system.

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y + z = -2 \\ 2x - 5y + 5z = 17 \end{cases}$$

#### Solution

In Example 4, Gaussian elimination was used to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 4 & \vdots & 7 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}.$$

Now, rather than using back-substitution, apply additional elementary row operations until you obtain a matrix in *reduced* row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$\begin{array}{r} 2R_2 + R_1 \rightarrow \\ \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 11 & \vdots & 23 \\ 0 & 1 & 4 & \vdots & 7 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Perform operations on $R_1$ so second column has a zero above its leading 1.

$$\begin{array}{r} -11R_3 + R_1 \rightarrow \\ -4R_3 + R_2 \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Perform operations on $R_1$ and $R_2$ so third column has zeros above its leading 1.

The matrix is now in reduced row-echelon form.
Converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1. \\ z = 2 \end{cases}$$

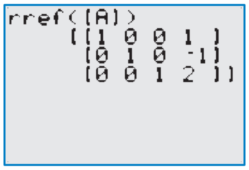
Now you can simply read the solution,

$$x = 1, \quad y = -1, \quad \text{and} \quad z = 2$$

which can be written as the ordered triple

$$(1, -1, 2).$$

You can check this result using the *reduced row-echelon form* feature of a graphing utility, as shown in Figure 7.23.



```
rref([A])
    [[1 0 0 1 ]
     [0 1 0 -1]
     [0 0 1 2 ]]
```

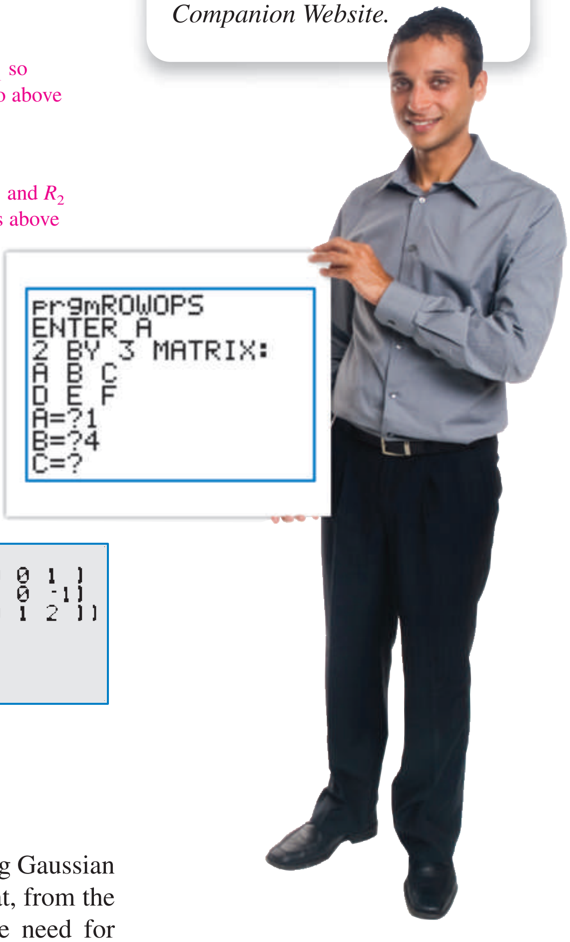
**Figure 7.23**

**CHECKPOINT** Now try Exercise 63.

In Example 8, note that the solution is the same as the one obtained using Gaussian elimination in Example 4. The advantage of Gauss-Jordan elimination is that, from the reduced row-echelon form, you can simply read the solution without the need for back-substitution.

### Technology Tip

For a demonstration of a graphical approach to Gauss-Jordan elimination on a 2 × 3 matrix, see the Visualizing Row Operations Program, available for several models of graphing calculators at this textbook's *Companion Website.*



```
prgmROWOPS
ENTER A
2 BY 3 MATRIX:
A B C
D E F
A=?1
B=?4
C=?
```

David Gilder/iStockphoto.com

The elimination procedures described in this section employ an algorithmic approach that is easily adapted to computer programs. However, the procedure makes no effort to avoid fractional coefficients. For instance, in the elimination procedure for the system

$$\begin{cases} 2x - 5y + 5z = 17 \\ 3x - 2y + 3z = 11 \\ -3x + 3y \phantom{+ 3z} = -6 \end{cases}$$

you may be inclined to multiply the first row by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. For hand computations, you can sometimes avoid fractions by judiciously choosing the order in which you apply elementary row operations.

## Example 9 A System with an Infinite Number of Solutions

Solve the system.

$$\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y \phantom{- 2z} = 1 \end{cases}$$

### Solution

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\begin{matrix} \\ -3R_1 + R_2 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix}$$

$$\begin{matrix} \\ -R_2 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$\begin{matrix} -2R_2 + R_1 \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x + 5z = 2 \\ y - 3z = -1 \end{cases}.$$

Solving for $x$ and $y$ in terms of $z$, you have

$$x = -5z + 2 \quad \text{and} \quad y = 3z - 1.$$

To write a solution of the system that does not use any of the three variables of the system, let $a$ represent any real number and let $z = a$. Now substitute $a$ for $z$ in the equations for $x$ and $y$.

$$x = -5z + 2 = -5a + 2$$

$$y = 3z - 1 = 3a - 1$$

So, the solution set has the form

$$(-5a + 2, 3a - 1, a).$$

Recall from Section 7.3 that a solution set of this form represents an infinite number of solutions. Try substituting values for $a$ to obtain a few solutions. Then check each solution in the original system of equations.

✓CHECKPOINT Now try Exercise 65.

## 7.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1–3, fill in the blank.**

**1.** A rectangular array of real numbers that can be used to solve a system of linear equations is called a _______ .

**2.** A matrix in row-echelon form is in _______ when every column that has a leading 1 has zeros in every position above and below its leading 1.

**3.** The process of using row operations to write a matrix in reduced row-echelon form is called _______ .

**In Exercises 4–6, refer to the system of linear equations** $\begin{cases} -2x + 3y = 5 \\ 6x + 7y = 4 \end{cases}$.

**4.** Is the coefficient matrix for the system a *square* matrix?

**5.** Is the augmented matrix for the system of dimension 3 × 2?

**6.** Is the augmented matrix row-equivalent to its reduced row-echelon form?

### Procedures and Problem Solving

**Dimension of a Matrix** **In Exercises 7–12, determine the dimension of the matrix.**

**7.** $\begin{bmatrix} 7 & 0 \end{bmatrix}$

**8.** $\begin{bmatrix} 3 & -1 & 2 & 6 \end{bmatrix}$

✓ **9.** $\begin{bmatrix} 4 \\ 32 \\ 3 \end{bmatrix}$

**10.** $\begin{bmatrix} 5 & 4 & 2 \\ 3 & -5 & 1 \\ 7 & -2 & 9 \end{bmatrix}$

**11.** $\begin{bmatrix} 33 & 45 \\ -9 & 20 \end{bmatrix}$

**12.** $\begin{bmatrix} 3 & -1 & 6 & 4 \\ -2 & 5 & 7 & 7 \end{bmatrix}$

**Writing an Augmented Matrix** **In Exercises 13–18, write the augmented matrix for the system of linear equations. What is the dimension of the augmented matrix?**

**13.** $\begin{cases} 4x - 3y = -5 \\ -x + 3y = 12 \end{cases}$

**14.** $\begin{cases} 7x + 4y = 22 \\ 5x - 9y = 15 \end{cases}$

✓ **15.** $\begin{cases} x + 10y - 2z = 2 \\ 5x - 3y + 4z = 0 \\ 2x + y = 6 \end{cases}$

**16.** $\begin{cases} x - 3y + z = 1 \\ 4y = 0 \\ 7z = -5 \end{cases}$

**17.** $\begin{cases} 7x - 5y + z = 13 \\ 19x - 8z = 10 \end{cases}$

**18.** $\begin{cases} 9x + 2y - 3z = 20 \\ -25y + 11z = -5 \end{cases}$

**Writing a System of Equations** **In Exercises 19–22, write the system of linear equations represented by the augmented matrix. (Use the variables *x*, *y*, *z*, and *w*, if applicable.)**

**19.** $\begin{bmatrix} 3 & 4 & \vdots & 9 \\ 1 & -1 & \vdots & -3 \end{bmatrix}$

**20.** $\begin{bmatrix} 7 & -5 & \vdots & 0 \\ 8 & 3 & \vdots & -2 \end{bmatrix}$

**21.** $\begin{bmatrix} 9 & 12 & 3 & \vdots & 0 \\ -2 & 18 & 5 & \vdots & 10 \\ 1 & 7 & -8 & \vdots & -4 \end{bmatrix}$

**22.** $\begin{bmatrix} 6 & 2 & -1 & -5 & \vdots & -25 \\ -1 & 0 & 7 & 3 & \vdots & 7 \\ 4 & -1 & -10 & 6 & \vdots & 23 \\ 0 & 8 & 1 & -11 & \vdots & -21 \end{bmatrix}$

**Identifying an Elementary Row Operation** **In Exercises 23–26, identify the elementary row operation performed to obtain the new row-equivalent matrix.**

| | *Original Matrix* | *New Row-Equivalent Matrix* |
|---|---|---|
| **23.** | $\begin{bmatrix} -3 & 6 & 0 \\ 5 & 2 & -2 \end{bmatrix}$ | $\begin{bmatrix} -18 & 0 & 6 \\ 5 & 2 & -2 \end{bmatrix}$ |
| **24.** | $\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$ | $\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$ |
| ✓ **25.** | $\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$ | $\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 4 & -5 & 1 & 3 \end{bmatrix}$ |
| **26.** | $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$ | $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 0 & -6 & 8 & -4 \end{bmatrix}$ |

**Elementary Row Operations** **In Exercises 27–30, fill in the blanks using elementary row operations to form a row-equivalent matrix.**

**27.** $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 10 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 & 3 \\ 0 & \square & -1 \end{bmatrix}$

**28.** $\begin{bmatrix} 3 & 6 & 8 \\ 4 & -3 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & \square & \frac{8}{3} \\ 4 & -3 & 6 \end{bmatrix}$

**29.** $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 3 & 8 & 10 & 3 \\ -2 & 1 & 12 & 6 \end{bmatrix}$ **30.** $\begin{bmatrix} 2 & 4 & 8 & 3 \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 5 & \square & \square \\ 0 & 3 & \square & \square \end{bmatrix}$ $\begin{bmatrix} 1 & \square & \square & \square \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 3 & \square & \square \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 4 & \frac{3}{2} \\ 0 & \square & -7 & \frac{1}{2} \\ 0 & 2 & \square & \square \end{bmatrix}$

**Comparing Linear Systems and Matrix Operations** **In Exercises 31 and 32, (a) perform the row operations to solve the augmented matrix, (b) write and solve the system of linear equations represented by the augmented matrix, and (c) compare the two solution methods. Which do you prefer?**

✓ **31.** $\begin{bmatrix} -3 & 4 & \vdots & 22 \\ 6 & -4 & \vdots & -28 \end{bmatrix}$

(i) Add $R_2$ to $R_1$.

(ii) Add $-2$ times $R_1$ to $R_2$.

(iii) Multiply $R_2$ by $-\frac{1}{4}$.

(iv) Multiply $R_1$ by $\frac{1}{3}$.

**32.** $\begin{bmatrix} 7 & 13 & 1 & \vdots & -4 \\ -3 & -5 & -1 & \vdots & -4 \\ 3 & 6 & 1 & \vdots & -2 \end{bmatrix}$

(i) Add $R_2$ to $R_1$.

(ii) Multiply $R_1$ and $\frac{1}{4}$.

(iii) Add $R_3$ to $R_2$.

(iv) Add $-3$ times $R_1$ to $R_3$.

(v) Add $-2$ times $R_2$ to $R_1$.

**33.** Repeat steps (i) through (iv) for the matrix in Exercise 31 using a graphing utility.

**34.** Repeat steps (i) through (v) for the matrix in Exercise 32 using a graphing utility.

**Row-Echelon Form** **In Exercises 35–40, determine whether the matrix is in row-echelon form. If it is, determine if it is also in reduced row-echelon form.**

✓ **35.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **36.** $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**37.** $\begin{bmatrix} 3 & 0 & 3 & 7 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ **38.** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

**39.** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ **40.** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

**Using Gaussian Elimination** **In Exercises 41–44, write the matrix in row-echelon form. Remember that the row-echelon form of a matrix is not unique.**

**41.** $\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & 4 & 0 & -5 \\ 2 & 6 & 3 & 10 \end{bmatrix}$

**42.** $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$

**43.** $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 5 & -4 & 1 & 8 \\ -6 & 8 & 18 & 0 \end{bmatrix}$

**44.** $\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 4 & -10 & 2 & -24 \end{bmatrix}$

**Using a Graphing Utility** **In Exercises 45–48, use the matrix capabilities of a graphing utility to write the matrix in reduced row-echelon form.**

**45.** $\begin{bmatrix} 3 & 3 & 3 \\ -1 & 0 & -4 \\ 2 & 4 & -2 \end{bmatrix}$ **46.** $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 15 & 9 \\ 2 & 6 & 10 \end{bmatrix}$

**47.** $\begin{bmatrix} -4 & 1 & 0 & 6 \\ 1 & -2 & 3 & -4 \end{bmatrix}$

**48.** $\begin{bmatrix} 5 & 1 & 2 & 4 \\ -1 & 5 & 10 & -32 \end{bmatrix}$

**Using Back-Substitution** **In Exercises 49–52, write the system of linear equations represented by the augmented matrix. Then use back-substitution to find the solution. (Use the variables $x$, $y$, and $z$, if applicable.)**

**49.** $\begin{bmatrix} 1 & -2 & \vdots & 4 \\ 0 & 1 & \vdots & -3 \end{bmatrix}$

**50.** $\begin{bmatrix} 1 & 8 & \vdots & 12 \\ 0 & 1 & \vdots & 3 \end{bmatrix}$

**51.** $\begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$

**52.** $\begin{bmatrix} 1 & 2 & -2 & \vdots & -1 \\ 0 & 1 & 1 & \vdots & 9 \\ 0 & 0 & 1 & \vdots & -3 \end{bmatrix}$

**Interpreting Reduced Row-Echelon Form** **In Exercises 53–56, an augmented matrix that represents a system of linear equations (in the variables $x$ and $y$ or $x$, $y$, and $z$) has been reduced using Gauss-Jordan elimination. Write the solution represented by the augmented matrix.**

**53.** $\begin{bmatrix} 1 & 0 & \vdots & 7 \\ 0 & 1 & \vdots & -5 \end{bmatrix}$ **54.** $\begin{bmatrix} 1 & 0 & \vdots & -2 \\ 0 & 1 & \vdots & 4 \end{bmatrix}$

**55.** $\begin{bmatrix} 1 & 0 & 0 & \vdots & -4 \\ 0 & 1 & 0 & \vdots & -8 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$

**56.** $\begin{bmatrix} 1 & 0 & 0 & \vdots & 3 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$

**Gaussian Elimination with Back-Substitution** **In Exercises 57–62, use matrices to solve the system of equations, if possible. Use Gaussian elimination with back-substitution.**

**57.** $\begin{cases} x + 2y = 7 \\ 2x + y = 8 \end{cases}$

**58.** $\begin{cases} 2x + 6y = 16 \\ 2x + 3y = 7 \end{cases}$

✓ **59.** $\begin{cases} -x + y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases}$

**60.** $\begin{cases} x + 2y = 0 \\ x + y = 6 \\ 3x - 2y = 8 \end{cases}$

✓ **61.** $\begin{cases} 3x + 2y - z + w = 0 \\ x - y + 4z + 2w = 25 \\ -2x + y + 2z - w = 2 \\ x + y + z + w = 6 \end{cases}$

**62.** $\begin{cases} x - 4y + 3z - 2w = 9 \\ 3x - 2y + z - 4w = -13 \\ -4x + 3y - 2z + w = -4 \\ -2x + y - 4z + 3w = -10 \end{cases}$

**Gauss-Jordan Elimination** **In Exercises 63–68, use matrices to solve the system of equations, if possible. Use Gauss-Jordan elimination.**

✓ **63.** $\begin{cases} x - 3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases}$

**64.** $\begin{cases} 2x - y + 3z = 24 \\ 2y - z = 14 \\ 7x - 5y = 6 \end{cases}$

✓ **65.** $\begin{cases} x + y - 5z = 3 \\ x - 2z = 1 \\ 2x - y - z = 0 \end{cases}$

**66.** $\begin{cases} 2x + 3z = 3 \\ 4x - 3y + 7z = 5 \\ 8x - 9y + 15z = 9 \end{cases}$

**67.** $\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$

**68.** $\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = 28 \\ -x + y = 14 \end{cases}$

**Using a Graphing Utility** **In Exercises 69–72, use the matrix capabilities of a graphing utility to reduce the augmented matrix corresponding to the system of equations, and solve the system.**

**69.** $\begin{cases} 3x + 3y + 12z = 6 \\ x + y + 4z = 2 \\ 2x + 5y + 20z = 10 \\ -x + 2y + 8z = 4 \end{cases}$

**70.** $\begin{cases} x + y + z = 0 \\ 2x + 3y + z = 0 \\ 3x + 5y + z = 0 \end{cases}$

**71.** $\begin{cases} 2x + 10y + 2z = 6 \\ x + 5y + 2z = 6 \\ x + 5y + z = 3 \\ -3x - 15y - 3z = -9 \end{cases}$

**72.** $\begin{cases} 2x + y - z + 2w = -6 \\ 3x + 4y + w = 1 \\ x + 5y + 2z + 6w = -3 \\ 5x + 2y - z - w = 3 \end{cases}$

**Comparing Solutions of Two Systems** **In Exercises 73–76, determine whether the two systems of linear equations yield the same solution. If so, find the solution.**

**73.** (a) $\begin{cases} x - 2y + z = -6 \\ y - 5z = 16 \\ z = -3 \end{cases}$ (b) $\begin{cases} x + y - 2z = 6 \\ y + 3z = -8 \\ z = -3 \end{cases}$

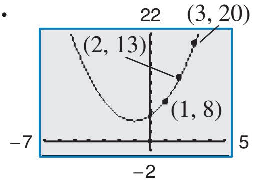
**74.** (a) $\begin{cases} x - 3y + 4z = -11 \\ y - z = -4 \\ z = 2 \end{cases}$ (b) $\begin{cases} x + 4y = -11 \\ y + 3z = 4 \\ z = 2 \end{cases}$

**75.** (a) $\begin{cases} x - 4y + 5z = 27 \\ y - 7z = -54 \\ z = 8 \end{cases}$ (b) $\begin{cases} x - 6y + z = 15 \\ y + 5z = 42 \\ z = 8 \end{cases}$

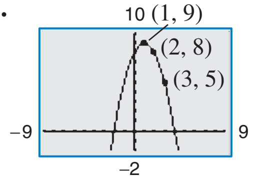
**76.** (a) $\begin{cases} x + 3y - z = 19 \\ y + 6z = -18 \\ z = -4 \end{cases}$ (b) $\begin{cases} x - y + 3z = -15 \\ y - 2z = 14 \\ z = -4 \end{cases}$

**Data Analysis: Curve Fitting** **In Exercises 77–80, use a system of equations to find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. Solve the system using matrices. Use a graphing utility to verify your result.**

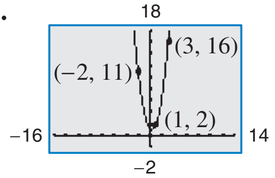
**77.**



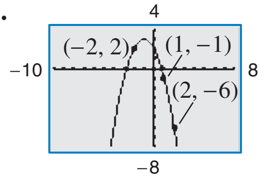
**78.**



**79.**



**80.**



**Curve Fitting In Exercises 81 and 82, use a system of equations to find the quadratic function $f(x) = ax^2 + bx + c$ that satisfies the equations. Solve the system using matrices.**

**81.** $f(-2) = -15$
$f(-1) = 7$
$f(1) = -3$

**82.** $f(-2) = -3$
$f(1) = -3$
$f(2) = -11$

**Curve Fitting In Exercises 83 and 84, use a system of equations to find the cubic function $f(x) = ax^3 + bx^2 + cx + d$ that satisfies the equations. Solve the system using matrices.**

**83.** $f(-2) = -7$
$f(-1) = 2$
$f(1) = -4$
$f(2) = -7$

**84.** $f(-2) = -17$
$f(-1) = -5$
$f(1) = 1$
$f(2) = 7$

**85. Electrical Engineering** The currents in an electrical network are given by the solution of the system

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 2I_1 + 2I_2 = 7 \\ 2I_2 + 4I_3 = 8 \end{cases}$$

where $I_1$, $I_2$, and $I_3$ are measured in amperes. Solve the system of equations using matrices.

**86. Finance** A corporation borrowed $1,500,000 to expand its line of shoes. Some of the money was borrowed at 3%, some at 4%, and some at 6%. Use a system of equations to determine how much was borrowed at each rate given that the annual interest was $74,000 and the amount borrowed at 6% was four times the amount borrowed at 3%. Solve the system using matrices.

**87. Using Matrices** A food server examines the amount of money earned in tips after working an 8-hour shift. The server has a total of $95 in denominations of $1, $5, $10, and $20 bills. The total number of paper bills is 26. The number of $5 bills is 4 times the number of $10 bills, and the number of $1 bills is 1 less than twice the number of $5 bills. Write a system of linear equations to represent the situation. Then use matrices to find the number of each denomination.

**88. Marketing** A wholesale paper company sells a 100-pound package of computer paper that consists of three grades, glossy, semi-gloss, and matte, for printing photographs. Glossy costs $5.50 per pound, semi-gloss costs $4.25 per pound, and matte costs $3.75 per pound. One half of the 100-pound package consists of the two less expensive grades. The cost of the 100-pound package is $480. Set up and solve a system of equations, using matrices, to find the number of pounds of each grade of paper in a 100-pound package.

Derek Latta/iStockphoto.com

**89. Partial Fractions** Use a system of equations to write the partial fraction decomposition of the rational expression. Solve the system using matrices.

$$\frac{8x^2}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

### 90. MODELING DATA

A video of the path of a ball thrown by a baseball player was analyzed with a grid covering the TV screen. The video was paused three times, and the position of the ball was measured each time. The coordinates obtained are shown in the table ($x$ and $y$ are measured in feet).

| Horizontal distance, $x$ | Height, $y$ |
|---|---|
| 0 | 5.0 |
| 15 | 9.6 |
| 30 | 12.4 |

(a) Use a system of equations to find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. Solve the system using matrices.

(b) Use a graphing utility to graph the parabola.

(c) Graphically approximate the maximum height of the ball and the point at which the ball strikes the ground.

(d) Algebraically approximate the maximum height of the ball and the point at which the ball strikes the ground.

**91.** ***Why you should learn it*** *(p. 504)* The table shows the average retail prices $y$ (in dollars) of prescriptions from 2006 through 2008. (Source: National Association of Chain Drug Stores)

| Year | Price, $y$ (in dollars) |
|---|---|
| 2006 | 65.82 |
| 2007 | 68.77 |
| 2008 | 71.70 |

(a) Use a system of equations to find the equation of the parabola $y = at^2 + bt + c$ that passes through the points. Let $t$ represent the year, with $t = 6$ corresponding to 2006. Solve the system using matrices.

(b) Use a graphing utility to graph the parabola and plot the data points.

(c) Use the equation in part (a) to estimate the average retail prices in 2010, 2015, and 2020.

(d) Are your estimates in part (c) reasonable? Explain.

**92. MODELING DATA**

The table shows the average annual salaries $y$ (in thousands of dollars) for public school classroom teachers in the United States from 2006 through 2008. (Source: Educational Research Service)

| Year | Annual salary, $y$ (in thousands of dollars) |
|---|---|
| 2006 | 48.2 |
| 2007 | 49.3 |
| 2008 | 51.3 |

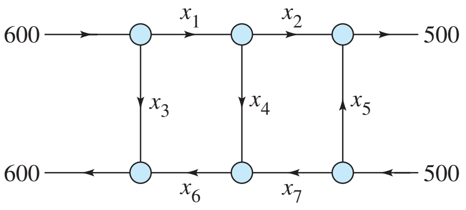
(a) Use a system of equations to find the equation of the parabola $y = at^2 + bt + c$ that passes through the points. Let $t$ represent the year, with $t = 6$ corresponding to 2006. Solve the system using matrices.

(b) Use a graphing utility to graph the parabola and plot the data points.

(c) Use the equation in part (a) to estimate the average annual salaries in 2010, 2015, and 2020.

(d) Are your estimates in part (c) reasonable? Explain.

**93. Network Analysis** Water flowing through a network of pipes (in thousands of cubic meters per hour) is shown below.



(a) Use matrices to solve this system for the water flow represented by $x_i$, $i = 1, 2, 3, 4, 5, 6$, and 7.

(b) Find the network flow pattern when $x_6 = 0$ and $x_7 = 0$.

(c) Find the network flow pattern when $x_5 = 400$ and $x_6 = 500$.

**94. Network Analysis** The flow of water (in thousands of cubic meters per hour) into and out of the right side of the network of pipes in Exercise 93 is increased from 500 to 700.

(a) Draw a diagram of the new network.

(b) Use matrices to solve this system for the water flow represented by $x_i$, $i = 1, 2, 3, 4, 5, 6$, and 7.

(c) Find the network flow pattern when $x_1 = 600$ and $x_7 = 200$.

(d) Find the network flow pattern when $x_4 = 150$ and $x_5 = 350$.

## Conclusions

**True or False?** **In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.**

**95.** When using Gaussian elimination to solve a system of linear equations, you may conclude that the system is inconsistent before you complete the process of rewriting the augmented matrix in row-echelon form.

**96.** You cannot write an augmented matrix for a dependent system of linear equations in reduced row-echelon form.

**97. Think About It** The augmented matrix represents a system of linear equations (in the variables $x$, $y$, and $z$) that has been reduced using Gauss-Jordan elimination. Write a system of three equations in three variables with *nonzero* coefficients that is represented by the reduced matrix. (There are many correct answers.)

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

**98. Error Analysis** Describe the errors.

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 3 & 5 \end{array}\right]$$

$$\xcancel{\begin{array}{l} -2R_1 + R_2 \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 5 \end{array}\right] \\ -R_2 + R_1 \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \end{array}\right] \end{array}}$$

**99. Think About It** Can a $2 \times 4$ augmented matrix whose entries are all nonzero real numbers represent an independent system of linear equations? Explain.

**100. CAPSTONE** Determine all values of $a$ and $b$ for which the augmented matrix has each given number of solutions.

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & a & b \end{array}\right]$$

(a) Exactly one solution

(b) Infinitely many solutions

(c) No solution

## Cumulative Mixed Review

**Graphing a Rational Function** **In Exercises 101–104, sketch the graph of the function. Identify any asymptotes.**

**101.** $f(x) = \dfrac{7}{-x - 1}$

**102.** $f(x) = \dfrac{4x}{5x^2 + 2}$

**103.** $f(x) = \dfrac{x^2 - 2x - 3}{x - 4}$

**104.** $f(x) = \dfrac{x^2 - 36}{x + 1}$

# 7.5 Operations with Matrices

### What you should learn

- Decide whether two matrices are equal.
- Add and subtract matrices and multiply matrices by scalars.
- Multiply two matrices.
- Use matrix operations to model and solve real-life problems.

### Why you should learn it

Matrix algebra provides a systematic way of performing mathematical operations on large arrays of numbers. In Exercise 89 on page 530, you will use matrix multiplication to help analyze the labor and wage requirements for a boat manufacturer.

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## Equality of Matrices

In Section 7.4, you used matrices to solve systems of linear equations. There is a rich mathematical theory of matrices, and its applications are numerous. This section and the next two introduce some fundamentals of matrix theory. It is standard mathematical convention to represent matrices in any of the following three ways.

### Representation of Matrices

**1.** A matrix can be denoted by an uppercase letter such as

$A$, $B$, or $C$.

**2.** A matrix can be denoted by a representative element enclosed in brackets, such as

$[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.

**3.** A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Two matrices

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

are **equal** when they have the same dimension $(m \times n)$ and all of their corresponding entries are equal.

### Example 1 Equality of Matrices

Solve for $a_{11}$, $a_{12}$, $a_{21}$, and $a_{22}$ in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

### Solution

Because two matrices are equal only when their corresponding entries are equal, you can conclude that

$$a_{11} = 2, \quad a_{12} = -1, \quad a_{21} = -3, \quad \text{and} \quad a_{22} = 0.$$

✓CHECKPOINT Now try Exercise 9.

Be sure you see that for two matrices to be equal, they must have the same dimension *and* their corresponding entries must be equal. For instance,

$$\begin{bmatrix} 2 & -1 \\ \sqrt{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0.5 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

## Matrix Addition and Scalar Multiplication

You can add two matrices (of the same dimension) by adding their corresponding entries.

### Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of dimension $m \times n$, then their sum is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different dimensions is undefined.

### Example 2 Addition of Matrices

**a.** $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 + 1 & 2 + 3 \\ 0 + (-1) & 1 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

**b.** $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**c.** The sum of

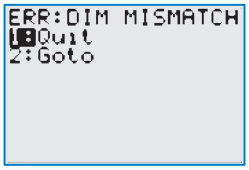
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

is undefined because $A$ is of dimension $2 \times 3$ and $B$ is of dimension $2 \times 2$.

CHECKPOINT Now try Exercise 15(a).

### Technology Tip

Try using a graphing utility to find the sum of the two matrices in Example 2(c). Your graphing utility should display an error message similar to the one shown below.



```
ERR:DIM MISMATCH
1:Quit
2:Goto
```

In operations with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. You can multiply a matrix $A$ by a scalar $c$ by multiplying each entry in $A$ by $c$.

### Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $c$ is a scalar, then the **scalar multiple** of $A$ by $c$ is the $m \times n$ matrix given by

$$cA = [ca_{ij}].$$

### Example 3 Scalar Multiplication

For the following matrix, find $3A$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

#### Solution

$$3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

CHECKPOINT Now try Exercise 15(c).

The symbol $-A$ represents the negation of $A$, which is the scalar product $(-1)A$. Moreover, if $A$ and $B$ are of the same dimension, then $A - B$ represents the sum of $A$ and $(-1)B$. That is,

$$A - B = A + (-1)B. \qquad \text{Subtraction of matrices}$$

The order of operations for matrix expressions is similar to that for real numbers. In particular, you perform scalar multiplication before matrix addition and subtraction, as shown in Example 4.

## Explore the Concept

What do you observe about the relationship between the corresponding entries of $A$ and $B$ below? Use a graphing utility to find $A + B$. What conclusion can you make about the entries of $A$ and $B$ and the sum $A + B$?

$$A = \begin{bmatrix} -1 & 5 \\ 2 & -6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -5 \\ -2 & 6 \end{bmatrix}$$

### Example 4 Scalar Multiplication and Matrix Subtraction

For the following matrices, find $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

### Solution

Note that $A$ is the same matrix from Example 3, where you found $3A$.

$$3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \qquad \text{Perform scalar multiplication first.}$$

$$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix} \qquad \text{Subtract corresponding entries.}$$

✓CHECKPOINT Now try Exercise 15(d).

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers. One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number $c$. For matrices, a similar property holds. That is, if $A$ is an $m \times n$ matrix and $O$ is the $m \times n$ **zero matrix** consisting entirely of zeros, then $A + O = A$.

In other words, $O$ is the **additive identity** for the set of all $m \times n$ matrices. For example, the following matrices are the additive identities for the sets of all $2 \times 3$ and $2 \times 2$ matrices.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

### Properties of Matrix Addition and Scalar Multiplication

Let $A$, $B$, and $C$ be $m \times n$ matrices and let $c$ and $d$ be scalars.

| | |
|---|---|
| **1.** $A + B = B + A$ | Commutative Property of Matrix Addition |
| **2.** $A + (B + C) = (A + B) + C$ | Associative Property of Matrix Addition |
| **3.** $(cd)A = c(dA)$ | Associative Property of Scalar Multiplication |
| **4.** $1A = A$ | Scalar Identity |
| **5.** $A + O = A$ | Additive Identity |
| **6.** $c(A + B) = cA + cB$ | Distributive Property |
| **7.** $(c + d)A = cA + dA$ | Distributive Property |

## Study Tip

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity because the same sum occurs no matter how the matrices are grouped. This same reasoning applies to sums of four or more matrices.

## Example 5 Using the Distributive Property

$$3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) = 3\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix}$$

CHECKPOINT Now try Exercise 23.

In Example 5, you could add the two matrices first and then multiply the resulting matrix by 3. The result would be the same.

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the following solutions.

| *Real Numbers*<br>*(Solve for x.)* | *$m \times n$ Matrices*<br>*(Solve for X.)* |
|---|---|
| $x + a = b$ | $X + A = B$ |
| $x + a + (-a) = b + (-a)$ | $X + A + (-A) = B + (-A)$ |
| $x + 0 = b - a$ | $X + O = B - A$ |
| $x = b - a$ | $X = B - A$ |

The algebra of real numbers and the algebra of matrices also have important differences, which will be discussed later.

## Example 6 Solving a Matrix Equation

Solve for $X$ in the equation

$$3X + A = B$$

where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

### Solution

Begin by solving the equation for $X$ to obtain

$$3X = B - A$$

$$X = \frac{1}{3}(B - A).$$

Now, using the matrices $A$ and $B$, you have

$$X = \frac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right) \qquad \text{Substitute the matrices.}$$

$$= \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} \qquad \text{Subtract matrix } A \text{ from matrix } B.$$

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. \qquad \text{Multiply the resulting matrix by } \tfrac{1}{3}.$$

CHECKPOINT Now try Exercise 31.

## Matrix Multiplication

Another basic matrix operation is **matrix multiplication.** At first glance, the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

### Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product $AB$ is an $m \times p$ matrix given by

$$AB = [c_{ij}]$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}$.

The definition of matrix multiplication indicates a *row-by-column* multiplication, where the entry in the $i$th row and $j$th column of the product $AB$ is obtained by multiplying the entries in the $i$th row of $A$ by the corresponding entries in the $j$th column of $B$ and then adding the results. The general pattern for matrix multiplication is as follows.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot\cdot\cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot\cdot\cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot\cdot\cdot & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdot\cdot\cdot & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdot\cdot\cdot & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdot\cdot\cdot & b_{1j} & \cdot\cdot\cdot & b_{1p} \\ b_{21} & b_{22} & \cdot\cdot\cdot & b_{2j} & \cdot\cdot\cdot & b_{2p} \\ b_{31} & b_{32} & \cdot\cdot\cdot & b_{3j} & \cdot\cdot\cdot & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdot\cdot\cdot & b_{nj} & \cdot\cdot\cdot & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdot\cdot\cdot & c_{1j} & \cdot\cdot\cdot & c_{1p} \\ c_{21} & c_{22} & \cdot\cdot\cdot & c_{2j} & \cdot\cdot\cdot & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdot\cdot\cdot & c_{ij} & \cdot\cdot\cdot & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdot\cdot\cdot & c_{mj} & \cdot\cdot\cdot & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj} = c_{ij}$$

## Example 7 Finding the Product of Two Matrices

Find the product $AB$ using $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

### Solution

First, note that the product $AB$ is defined because the number of columns of $A$ is equal to the number of rows of $B$. Moreover, the product $AB$ has dimension $3 \times 2$. To find the entries of the product, multiply each row of $A$ by each column of $B$.

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix} \\ &= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix} \end{aligned}$$

CHECKPOINT Now try Exercise 35.

Be sure you understand that for the product of two matrices to be defined, the number of *columns* of the first matrix must equal the number of *rows* of the second matrix. That is, the middle two indices must be the same. The outside two indices give the dimension of the product, as shown in the following diagram.

$$\underset{m \times n}{A} \quad \times \quad \underset{n \times p}{B} \quad = \quad \underset{m \times p}{AB}$$

Equal

Dimension of $AB$

## Example 8 Matrix Multiplication

**a.** $\underset{2 \times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \underset{3 \times 3}{\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}} = \underset{2 \times 3}{\begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}}$

**b.** $\underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$

**c.** $\underset{2 \times 2}{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

**d.** $\underset{3 \times 3}{\begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} 10 \\ -5 \\ -9 \end{bmatrix}}$

**e.** The product $AB$ for the following matrices is not defined.

$$A = \underset{3 \times 2}{\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix}} \quad \text{and} \quad B = \underset{3 \times 4}{\begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}}$$

**CHECKPOINT** Now try Exercise 37.

## Example 9 Matrix Multiplication

**a.** $\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} 1 \end{bmatrix}}$ **b.** $\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} \underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} = \underset{3 \times 3}{\begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}}$

**CHECKPOINT** Now try Exercise 45.

In Example 9, note that the two products are different. Even when both $AB$ and $BA$ are defined, matrix multiplication is not, in general, commutative. That is, for most matrices,

$$AB \neq BA.$$

This is one way in which the algebra of real numbers and the algebra of matrices differ.

### Explore the Concept

Use the following matrices to find $AB$, $BA$, $(AB)C$, and $A(BC)$. What do your results tell you about matrix multiplication and commutativity and associativity?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

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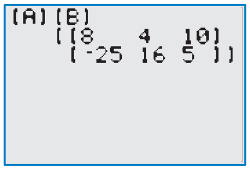
## Example 10 Matrix Multiplication

Use a graphing utility to find the product $AB$ using

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -5 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & -2 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

### Solution

Note that the dimension of $A$ is $2 \times 3$ and the dimension of $B$ is $3 \times 3$. So, the product will have dimension $2 \times 3$. Use the *matrix editor* to enter $A$ and $B$ into the graphing utility. Then, find the product, as shown in Figure 7.24.



```
[A][B]
    [[8    4   10]
     [-25 16   5 ]]
```

**Figure 7.24**

**CHECKPOINT** Now try Exercise 47.

### Properties of Matrix Multiplication

Let $A$, $B$, and $C$ be matrices and let $c$ be a scalar.

| | |
|---|---|
| **1.** $A(BC) = (AB)C$ | Associative Property of Matrix Multiplication |
| **2.** $A(B + C) = AB + AC$ | Left Distributive Property |
| **3.** $(A + B)C = AC + BC$ | Right Distributive Property |
| **4.** $c(AB) = (cA)B = A(cB)$ | Associative Property of Scalar Multiplication |

### Definition of Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of dimension $n \times n$** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \qquad \text{Identity matrix}$$

Note that an identity matrix must be *square*. When the dimension is understood to be $n \times n$, you can denote $I_n$ simply by $I$.

If $A$ is an $n \times n$ matrix, then the identity matrix has the property that $AI_n = A$ and $I_nA = A$. For example,

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} \qquad AI = A$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}. \qquad IA = A$$

## Applications

Matrix multiplication can be used to represent a system of linear equations. Note how the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

can be written as the matrix equation

$$AX = B$$

where $A$ is the *coefficient matrix* of the system, $B$ is the *constant matrix* of the system, and $X$ is a column matrix.

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{\times}{\phantom{x}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

### Example 11 Solving a System of Linear Equations

Consider the following system of linear equations.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ \quad\quad x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

**a.** Write this system as a matrix equation $AX = B$.

**b.** Use Gauss-Jordan elimination on $[A \vdots B]$ to solve for the matrix $X$.

#### Solution

**a.** In matrix form $AX = B$, the system can be written as follows.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

**b.** The augmented matrix is

$$[A \vdots B] = \left[\begin{array}{ccc:c} 1 & -2 & 1 & -4 \\ 0 & 1 & 2 & 4 \\ 2 & 3 & -2 & 2 \end{array}\right].$$

Using Gauss-Jordan elimination, you can rewrite this equation as

$$[I \vdots X] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

So, the solution of the system of linear equations is

$$x_1 = -1,\ x_2 = 2, \text{ and } x_3 = 1.$$

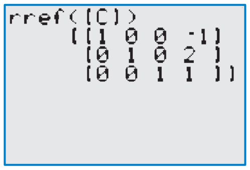
The solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

**CHECKPOINT** Now try Exercise 63.

### Technology Tip

Most graphing utilities can be used to obtain the reduced row-echelon form of a matrix. The screen below shows how one graphing utility displays the reduced row-echelon form of the augmented matrix in Example 11.



```
rref([C])
      [[1 0 0 -1]
       [0 1 0 2 ]
       [0 0 1 1 ]]
```

### Example 12 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors, as shown in the table.

| Equipment | Women's team | Men's team |
|---|---|---|
| Bats | 12 | 15 |
| Balls | 45 | 38 |
| Gloves | 15 | 17 |

The equipment costs are as follows.

*Bats:* \$90 per bat

*Balls:* \$6 per ball

*Gloves:* \$60 per glove

Use matrices to find the total cost of equipment for each team.

### Solution

The equipment list $E$ can be written in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}.$$

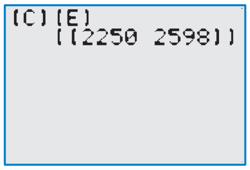
The costs per item $C$ can be written in matrix form as

$$C = [90 \quad 6 \quad 60].$$

You can find the total cost of the equipment for each team using the product $CE$ because the number of columns of $C$ (3 columns) equals the number of rows of $E$ (3 rows). Therefore, the total cost of equipment for each team is given by

$$\begin{aligned} CE &= [90 \quad 6 \quad 60]\begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \\ &= [90(12) + 6(45) + 60(15) \quad 90(15) + 6(38) + 60(17)] \\ &= [2250 \quad 2598]. \end{aligned}$$

So, the total cost of equipment for the women's team is \$2250, and the total cost of equipment for the men's team is \$2598. You can use a graphing utility to check this result, as shown in Figure 7.25.



```
[C][E]
   [[2250 2598]]
```

Figure 7.25

**CHECKPOINT** Now try Exercise 87.

Notice in Example 12 that you cannot find the total cost using the product $EC$ because $EC$ is not defined. That is, the number of columns of $E$ (2 columns) does not equal the number of rows of $C$ (1 row).

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## 7.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1–4, fill in the blank(s).**

**1.** Two matrices are _______ if all of their corresponding entries are equal.

**2.** When working with matrices, real numbers are often referred to as _______ .

**3.** A matrix consisting entirely of zeros is called a _______ matrix and is denoted by _______ .

**4.** The $n \times n$ matrix consisting of 1's on its main diagonal and 0's elsewhere is called the _______ matrix of dimension $n$.

**In Exercises 5 and 6, match the matrix property with the correct form. $A$, $B$, and $C$ are matrices, and $c$ and $d$ are scalars.**

**5.** (a) $(cd)A = c(dA)$ (i) Commutative Property of Matrix Addition
(b) $A + B = B + A$ (ii) Associative Property of Matrix Addition
(c) $1A = A$ (iii) Associative Property of Scalar Multiplication
(d) $c(A + B) = cA + cB$ (iv) Scalar Identity
(e) $A + (B + C) = (A + B) + C$ (v) Distributive Property

**6.** (a) $A(B + C) = AB + AC$ (i) Associative Property of Matrix Multiplication
(b) $c(AB) = (cA)B = A(cB)$ (ii) Left Distributive Property
(c) $A(BC) = (AB)C$ (iii) Right Distributive Property
(d) $(A + B)C = AC + BC$ (iv) Associative Property of Scalar Multiplication

**7.** In general, when multiplying matrices $A$ and $B$, does $AB = BA$?

**8.** What is the dimension of $AB$ given $A$ is a $2 \times 3$ matrix and $B$ is a $3 \times 4$ matrix?

### Procedures and Problem Solving

**Equality of Matrices** **In Exercises 9–12, find $x$ and $y$ or $x$, $y$, and $z$.**

✓ **9.** $\begin{bmatrix} x & -7 \\ 9 & y \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ 9 & -8 \end{bmatrix}$

**10.** $\begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$

**11.** $\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 12 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+7 & 4 \\ -3 & 13 & 15 & 3y \\ 0 & 2 & 3z-14 & 0 \end{bmatrix}$

**12.** $\begin{bmatrix} x+4 & 8 & -3 \\ 1 & 22 & 2y \\ 7 & -2 & z+2 \end{bmatrix} = \begin{bmatrix} 2x+9 & 8 & -3 \\ 1 & 22 & -8 \\ 7 & -2 & 11 \end{bmatrix}$

**Operations with Matrices** **In Exercises 13–20, find, if possible, (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$. Use the matrix capabilities of a graphing utility to verify your results.**

**13.** $A = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$

**14.** $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

✓ **15.** $A = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix}$

**16.** $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 & -5 \\ -3 & 4 & -7 \end{bmatrix}$

**17.** $A = \begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix}$

**18.** $A = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -2 & 2 \\ 5 & 4 & -1 \\ 0 & 8 & -6 \\ -4 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 & 1 \\ 2 & -4 & -7 \\ 10 & -9 & -1 \\ 3 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}$

**19.** $A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$

**20.** $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 6 & 2 \end{bmatrix}$

**Evaluating an Expression** In Exercises 21–24, evaluate the expression.

**21.** $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$

**22.** $\begin{bmatrix} 6 & 9 \\ -1 & 0 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -2 & -1 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} -13 & -7 \\ 4 & -1 \\ -6 & 0 \end{bmatrix}$

✓ **23.** $4\left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix}\right)$

**24.** $\frac{1}{2}(\begin{bmatrix} 5 & -2 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 6 & -18 & 9 \end{bmatrix})$

**Operations with Matrices** In Exercises 25–28, use the matrix capabilities of a graphing utility to evaluate the expression. Round your results to the nearest thousandths, if necessary.

**25.** $\frac{3}{7}\begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix} + 6\begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}$

**26.** $55\begin{bmatrix} 14 & -11 \\ -22 & 19 \end{bmatrix} + \begin{bmatrix} -22 & 20 \\ 13 & 6 \end{bmatrix}$

**27.** $-1\begin{bmatrix} 3.211 & 6.829 \\ -1.004 & 4.914 \\ 0.055 & -3.889 \end{bmatrix} - 1\begin{bmatrix} 1.630 & -3.090 \\ 5.256 & 8.335 \\ -9.768 & 4.251 \end{bmatrix}$

**28.** $-1\begin{bmatrix} 10 & 15 \\ -20 & 10 \\ 12 & 4 \end{bmatrix} + \frac{1}{8}\left(\begin{bmatrix} -13 & 11 \\ 7 & 0 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} -3 & 13 \\ -3 & 8 \\ -14 & 15 \end{bmatrix}\right)$

**Solving a Matrix Equation** In Exercises 29–32, solve for $X$ when

$$\mathbf{A} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}.$$

**29.** $X = 3A - 2B$

**30.** $2X = 2A - B$

✓ **31.** $2X + 3A = B$

**32.** $2A + 4B = -2X$

**Finding the Product of Two Matrices** In Exercises 33–40, find $AB$, if possible.

**33.** $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ -1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -3 & 0 \\ 4 & 0 & 2 \\ 8 & -2 & 7 \end{bmatrix}$

**34.** $A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$

✓ **35.** $A = \begin{bmatrix} -1 & 6 \\ -4 & 5 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 9 \end{bmatrix}$

**36.** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

✓ **37.** $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

**38.** $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -11 & 4 \\ 8 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

**39.** $A = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -1 & -5 & -9 \end{bmatrix}$

**40.** $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 6 & 13 & 8 & -17 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$

**Operations with Matrices** In Exercises 41–46, find, if possible, (a) $AB$, (b) $BA$, and (c) $A^2$. (*Note:* $A^2 = AA$.) Use the matrix capabilities of a graphing utility to verify your results.

**41.** $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

**42.** $A = \begin{bmatrix} 6 & 3 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$

**43.** $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

**44.** $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

✓ **45.** $A = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

**46.** $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

**Matrix Multiplication** In Exercises 47–50, use the matrix capabilities of a graphing utility to find $AB$, if possible.

✓ **47.** $A = \begin{bmatrix} 7 & 5 & -4 \\ -2 & 5 & 1 \\ 10 & -4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 3 \\ 8 & 1 & 4 \\ -4 & 2 & -8 \end{bmatrix}$

**48.** $A = \begin{bmatrix} 1 & -12 & 4 \\ 14 & 10 & 12 \\ 6 & -15 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 12 & 10 \\ -6 & 12 \\ 10 & 16 \end{bmatrix}$

**49.** $A = \begin{bmatrix} -3 & 8 & -6 & 8 \\ -12 & 15 & 9 & 6 \\ 5 & -1 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 6 \\ 24 & 15 & 14 \\ 16 & 10 & 21 \\ 8 & -4 & 10 \end{bmatrix}$

**50.** $A = \begin{bmatrix} -2 & 6 & 12 \\ 21 & -5 & 6 \\ 13 & -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ -7 & 18 \\ 34 & 14 \\ 0.5 & 1.4 \end{bmatrix}$

**Operations with Matrices** **In Exercises 51–54, use the matrix capabilities of a graphing utility to evaluate the expression.**

**51.** $\begin{bmatrix}3&1\\0&-2\end{bmatrix}\begin{bmatrix}1&0\\-2&2\end{bmatrix}\begin{bmatrix}1&0\\2&4\end{bmatrix}$

**52.** $-3\begin{bmatrix}6&5&-1\\1&-2&0\end{bmatrix}\begin{bmatrix}0&3\\-1&-3\\4&1\end{bmatrix}$

**53.** $\begin{bmatrix}0&2&-2\\4&1&2\end{bmatrix}\left(\begin{bmatrix}4&0\\0&-1\\-1&2\end{bmatrix}+\begin{bmatrix}-2&3\\-3&5\\0&-3\end{bmatrix}\right)$

**54.** $\begin{bmatrix}3\\-1\\5\\7\end{bmatrix}([5\quad -6]+[7\quad -1]+[-8\quad 9])$

**Matrix Multiplication** **In Exercises 55–58, use matrix multiplication to determine whether each matrix is a solution of the system of equations. Use a graphing utility to verify your results.**

**55.** $\begin{cases}x+2y=4\\3x+2y=0\end{cases}$

(a) $\begin{bmatrix}2\\1\end{bmatrix}$ (b) $\begin{bmatrix}-2\\3\end{bmatrix}$

(c) $\begin{bmatrix}-4\\4\end{bmatrix}$ (d) $\begin{bmatrix}2\\-3\end{bmatrix}$

**56.** $\begin{cases}6x+2y=0\\-x+5y=16\end{cases}$

(a) $\begin{bmatrix}-1\\3\end{bmatrix}$ (b) $\begin{bmatrix}2\\-6\end{bmatrix}$

(c) $\begin{bmatrix}3\\-9\end{bmatrix}$ (d) $\begin{bmatrix}-3\\9\end{bmatrix}$

**57.** $\begin{cases}-2x-3y=-6\\4x+2y=20\end{cases}$

(a) $\begin{bmatrix}3\\0\end{bmatrix}$ (b) $\begin{bmatrix}6\\-2\end{bmatrix}$

(c) $\begin{bmatrix}-6\\6\end{bmatrix}$ (d) $\begin{bmatrix}4\\2\end{bmatrix}$

**58.** $\begin{cases}5x-7y=-15\\3x+y=17\end{cases}$

(a) $\begin{bmatrix}4\\5\end{bmatrix}$ (b) $\begin{bmatrix}5\\2\end{bmatrix}$

(c) $\begin{bmatrix}-4\\-5\end{bmatrix}$ (d) $\begin{bmatrix}2\\11\end{bmatrix}$

**Solving a System of Linear Equations** **In Exercises 59–66, (a) write the system of equations as a matrix equation $AX = B$ and (b) use Gauss-Jordan elimination on the augmented matrix $[A \vdots B]$ to solve for the matrix $X$. Use a graphing utility to check your solution.**

**59.** $\begin{cases}-x_1+x_2=4\\-2x_1+x_2=0\end{cases}$

**60.** $\begin{cases}2x_1+3x_2=5\\x_1+4x_2=10\end{cases}$

**61.** $\begin{cases}-2x_1-3x_2=-4\\6x_1+x_2=-36\end{cases}$

**62.** $\begin{cases}-4x_1+9x_2=-13\\x_1-3x_2=12\end{cases}$

✓ **63.** $\begin{cases}x_1-2x_2+3x_3=9\\-x_1+3x_2-x_3=-6\\2x_1-5x_2+5x_3=17\end{cases}$

**64.** $\begin{cases}x_1+x_2-3x_3=9\\-x_1+2x_2=6\\x_1-x_2+x_3=-5\end{cases}$

**65.** $\begin{cases}x_1-5x_2+2x_3=-20\\-3x_1+x_2-x_3=8\\-2x_2+5x_3=-16\end{cases}$

**66.** $\begin{cases}x_1-x_2+4x_3=17\\x_1+3x_2=-11\\-6x_2+5x_3=40\end{cases}$

**Operations with Matrices** **In Exercises 67–72, use a graphing utility to perform the operations for the matrices $A$, $B$, and $C$, and the scalar $c$. Write a brief statement comparing the results of parts (a) and (b).**

$$A=\begin{bmatrix}1&2&-1\\0&-2&3\\4&-3&2\end{bmatrix},\quad B=\begin{bmatrix}2&3&0\\4&1&-2\\-1&2&0\end{bmatrix},$$

$$C=\begin{bmatrix}3&-2&1\\-4&0&3\\-1&3&-2\end{bmatrix},\quad \text{and}\quad c=3$$

**67.** (a) $A(B+C)$ (b) $AB+AC$

**68.** (a) $(B+C)A$ (b) $BA+CA$

**69.** (a) $(A+B)^2$ (b) $A^2+AB+BA+B^2$

**70.** (a) $(A-B)^2$ (b) $A^2-AB-BA+B^2$

**71.** (a) $A(BC)$ (b) $(AB)C$

**72.** (a) $c(AB)$ (b) $(cA)B$

**Operations with Matrices** **In Exercises 73–80, perform the operations (a) using a graphing utility and (b) by hand algebraically. If it is not possible to perform the operation(s), state the reason.**

$$A=\begin{bmatrix}1&2&-2\\-1&1&0\end{bmatrix},\quad B=\begin{bmatrix}-1&4&-1\\-2&-1&0\end{bmatrix},$$

$$C=\begin{bmatrix}1&2\\-2&3\\1&0\end{bmatrix},\quad c=-2,\quad \text{and}\quad d=-3$$

**73.** $A+cB$

**74.** $A(B+C)$

**75.** $c(AB)$

**76.** $B+dA$

**77.** $CA-BC$

**78.** $dAB^2$

**79.** $cdA$

**80.** $cA+dB$

**Using a Graphing Utility** **In Exercises 81 and 82, use the matrix capabilities of a graphing utility to find $f(A)=a_0I_n+a_1A+a_2A^2$.**

**81.** $f(x)=x^2-5x+2,\quad A=\begin{bmatrix}2&0\\4&5\end{bmatrix}$

**82.** $f(x)=x^2-7x+6,\quad A=\begin{bmatrix}5&4\\1&2\end{bmatrix}$

**83. Manufacturing** A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. The number of units of guitar $i$ produced at factory $j$ in one day is represented by $a_{ij}$ in the matrix

$$A = \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix}.$$

Find the production levels when production is increased by 20%.

**84. Manufacturing** A corporation has four factories, each of which manufactures sport utility vehicles and pickup trucks. The number of units of vehicle $i$ produced at factory $j$ in one day is represented by $a_{ij}$ in the matrix

$$A = \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix}.$$

Find the production levels when production is decreased by 10%.

**85. Manufacturing** A corporation that makes sunglasses has four factories, each of which manufactures two products. The number of units of product $i$ produced at factory $j$ in one day is represented by $a_{ij}$ in the matrix

$$A = \begin{bmatrix} 100 & 120 & 60 & 40 \\ 140 & 160 & 200 & 80 \end{bmatrix}.$$

Find the production levels when production is decreased by 10%.

**86. Tourism** A vacation service has identified four resort hotels with a special all-inclusive package (room and meals included) at a popular travel destination. The quoted room rates are for double and family (maximum of four people) occupancy for 5 days and 4 nights. The current rates for the two types of rooms at the four hotels are represented by matrix $A$.

$$A = \begin{array}{c} \begin{array}{cccc} \text{Hotel } w & \text{Hotel } x & \text{Hotel } y & \text{Hotel } z \end{array} \\ \begin{bmatrix} 615 & 670 & 740 & 990 \\ 995 & 1030 & 1180 & 1105 \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{l} \text{Double} \\ \text{Family} \end{array}\right\} \text{Occupancy} \end{array}$$

Room rates are guaranteed not to increase by more than 12%. What is the maximum rate per package per hotel during the next year?

✓ **87. Agriculture** A fruit grower raises two crops, apples and peaches. Each of these crops is shipped to three different outlets. The number of units of crop $i$ that are shipped to outlet $j$ is represented by $a_{ij}$ in the matrix

$$A = \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix}.$$

The profit per unit is represented by the matrix

$$B = [\$3.50 \quad \$6.00].$$

Find the product $BA$ and state what each entry of the product represents.

**88. Inventory Control** A company sells five models of computers through three retail outlets. The inventories are given by $S$. The wholesale and retail prices are given by $T$. Compute $ST$ and interpret the result.

$$S = \begin{array}{c} \overbrace{\begin{array}{ccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{array}}^{\text{Model}} \\ \begin{bmatrix} 3 & 2 & 2 & 3 & 0 \\ 0 & 2 & 3 & 4 & 3 \\ 4 & 2 & 1 & 3 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{l} 1 \\ 2 \\ 3 \end{array}\right\} \text{Outlet} \end{array}$$

$$T = \begin{array}{c} \overbrace{\begin{array}{cc} \text{Wholesale} & \text{Retail} \end{array}}^{\text{Price}} \\ \begin{bmatrix} \$840 & \$1100 \\ \$1200 & \$1350 \\ \$1450 & \$1650 \\ \$2650 & \$3000 \\ \$3050 & \$3200 \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}\right\} \text{Model} \end{array}$$

**89.** ***Why you should learn it*** *(p. 518)* A company that manufactures boats has the following labor-hour and wage requirements. Compute $ST$ and interpret the result.

Labor per Boat

$$S = \begin{array}{c} \overbrace{\begin{array}{ccc} \text{Cutting} & \text{Assembly} & \text{Packaging} \end{array}}^{\text{Department}} \\ \begin{bmatrix} 1.0 \text{ hr} & 0.5 \text{ hr} & 0.2 \text{ hr} \\ 1.6 \text{ hr} & 1.0 \text{ hr} & 0.2 \text{ hr} \\ 2.5 \text{ hr} & 2.0 \text{ hr} & 1.4 \text{ hr} \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{l} \text{Small} \\ \text{Medium} \\ \text{Large} \end{array}\right\} \begin{array}{l} \text{Boat} \\ \text{size} \end{array} \end{array}$$

Wages per Hour

$$T = \begin{array}{c} \overbrace{\begin{array}{cc} \text{A} & \text{B} \end{array}}^{\text{Plant}} \\ \begin{bmatrix} \$15 & \$13 \\ \$12 & \$11 \\ \$11 & \$10 \end{bmatrix} \end{array} \begin{array}{l} \\ \left.\begin{array}{l} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{array}\right\} \text{Department} \end{array}$$

**90. Physical Education** The numbers of calories burned by individuals of different weights performing different types of aerobic exercises for 20-minute time periods are shown in the matrix.

$$B = \begin{array}{c} \begin{array}{cc} \text{120-lb person} & \text{150-lb person} \end{array} \\ \begin{bmatrix} 109 & 136 \\ 127 & 159 \\ 64 & 79 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Bicycling} \\ \text{Jogging} \\ \text{Walking} \end{array}$$

(a) A 120-pound person and a 150-pound person bicycle for 40 minutes, jog for 10 minutes, and walk for 60 minutes. Organize a matrix $A$ for the time spent exercising in units of 20-minute intervals.

(b) Find the product $AB$.

(c) Explain the meaning of the product $AB$ in the context of the situation.

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**91.** **Politics** The matrix

$$\begin{array}{cc} & \overbrace{\begin{array}{ccc}\text{R} & \text{D} & \text{I}\end{array}}^{\text{From}} \\ P = & \left[\begin{array}{ccc}0.6 & 0.1 & 0.1\\ 0.2 & 0.7 & 0.1\\ 0.2 & 0.2 & 0.8\end{array}\right]\left.\begin{array}{c}1\\2\\3\end{array}\right\}\text{To}\end{array}$$

is called a *stochastic matrix*. Each entry $p_{ij}\,(i \neq j)$ represents the proportion of the voting population that changes from party $i$ to party $j$, and $p_{ii}$ represents the proportion that remains loyal to the party from one election to the next. Compute and interpret $P^2$.

**92.** **Politics** Use a graphing utility to find $P^3$, $P^4$, $P^5$, $P^6$, $P^7$, and $P^8$ for the matrix given in Exercise 91. Can you detect a pattern as $P$ is raised to higher powers?

## Conclusions

**True or False?** **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

**93.** Two matrices can be added only when they have the same dimension.

**94.** $\begin{bmatrix}-6 & -2\\ 2 & -6\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}4 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-6 & -2\\ 2 & -6\end{bmatrix}$

**Think About It** **In Exercises 95–102, let matrices $A$, $B$, $C$, and $D$ be of dimensions $2 \times 3$, $2 \times 3$, $3 \times 2$, and $2 \times 2$, respectively. Determine whether the matrices are of proper dimension to perform the operation(s). If so, give the dimension of the answer.**

**95.** $A + 2C$

**96.** $B - 3C$

**97.** $AB$

**98.** $BC$

**99.** $BC - D$

**100.** $CB - D$

**101.** $D(A - 3B)$

**102.** $(BC - D)A$

**Think About It** **In Exercises 103–106, use the matrices**

$$A = \begin{bmatrix}\mathbf{2} & \mathbf{-1}\\ \mathbf{1} & \mathbf{3}\end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix}\mathbf{-1} & \mathbf{1}\\ \mathbf{0} & \mathbf{-2}\end{bmatrix}.$$

**103.** Show that $(A + B)^2 \neq A^2 + 2AB + B^2$.

**104.** Show that $(A - B)^2 \neq A^2 - 2AB + B^2$.

**105.** Show that $(A + B)(A - B) \neq A^2 - B^2$.

**106.** Show that $(A + B)^2 = A^2 + AB + BA + B^2$.

**107.** **Think About It** If $a$, $b$, and $c$ are real numbers such that $c \neq 0$ and $ac = bc$, then $a = b$. However, if $A$, $B$, and $C$ are nonzero matrices such that $AC = BC$, then $A$ is *not necessarily* equal to $B$. Illustrate this using the following matrices.

$$A = \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}, \quad B = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \quad C = \begin{bmatrix}2 & 3\\ 2 & 3\end{bmatrix}$$

**108.** **Think About It** If $a$ and $b$ are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$. However, if $A$ and $B$ are matrices such that $AB = O$, it is *not necessarily* true that $A = O$ or $B = O$. Illustrate this using the following matrices.

$$A = \begin{bmatrix}3 & 3\\ 4 & 4\end{bmatrix}, \quad B = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

**109.** **Exploration** Let $i = \sqrt{-1}$ and let

$$A = \begin{bmatrix}i & 0\\ 0 & i\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}.$$

(a) Find $A^2$, $A^3$, and $A^4$. Identify any similarities with $i^2$, $i^3$, and $i^4$.

(b) Find and identify $B^2$.

**110.** **Think About It** Let $A$ and $B$ be unequal diagonal matrices of the same dimension. (A **diagonal matrix** is a square matrix in which each entry not on the main diagonal is zero.) Determine the products $AB$ for several pairs of such matrices. Make a conjecture about a quick rule for such products.

**111.** **Exploration** Consider matrices of the form

$$A = \begin{bmatrix}0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1n}\\ 0 & 0 & a_{23} & a_{24} & \cdots & a_{2n}\\ 0 & 0 & 0 & a_{34} & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & a_{(n-1)n}\\ 0 & 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

(a) Write a $2 \times 2$ matrix and a $3 \times 3$ matrix in the form of $A$.

(b) Use a graphing utility to raise each of the matrices to higher powers. Describe the result.

(c) Use the result of part (b) to make a conjecture about powers of $A$ when $A$ is a $4 \times 4$ matrix. Use the graphing utility to test your conjecture.

(d) Use the results of parts (b) and (c) to make a conjecture about powers of an $n \times n$ matrix $A$.

**112.** **CAPSTONE** Let matrices $A$ and $B$ be of dimensions $3 \times 2$ and $2 \times 2$, respectively. Answer the following questions and explain your reasoning.

(a) Is it possible that $A = B$?

(b) Is $A + B$ defined?

## Cumulative Mixed Review

**Condensing a Logarithmic Expression** **In Exercises 113 and 114, condense the expression to the logarithm of a single quantity.**

**113.** $3 \ln 4 - \frac{1}{3}\ln(x^2 + 3)$

**114.** $\frac{3}{2}\ln 7t^4 - \frac{3}{5}\ln t^5$

# 7.6 The Inverse of a Square Matrix

## The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation

$$ax = b.$$

To solve this equation for $x$, multiply each side of the equation by $a^{-1}$ (provided that $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number $a^{-1}$ is called the *multiplicative inverse* of $a$ because

$$a^{-1}a = 1.$$

The definition of the multiplicative **inverse of a matrix** is similar.

### Definition of the Inverse of a Square Matrix

Let $A$ be an $n \times n$ matrix and let $I_n$ be the $n \times n$ identity matrix. If there exists a matrix $A^{-1}$ such that

$$AA^{-1} = I_n = A^{-1}A$$

then $A^{-1}$ is called the **inverse** of $A$. The symbol $A^{-1}$ is read "$A$ inverse."

### What you should learn

- Verify that two matrices are inverses of each other.
- Use Gauss-Jordan elimination to find inverses of matrices.
- Use a formula to find inverses of $2 \times 2$ matrices.
- Use inverse matrices to solve systems of linear equations.

### Why you should learn it

A system of equations can be solved using the inverse of the coefficient matrix. This method is particularly useful when the coefficients are the same for several systems, but the constants are different. Exercise 66 on page 540 shows how to use an inverse matrix to find a model for the number of international travelers to the United States from Europe.

### Example 1 The Inverse of a Matrix

Show that $B$ is the inverse of $A$, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

### Solution

To show that $B$ is the inverse of $A$, show that $AB = I = BA$, as follows.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As you can see,

$$AB = I = BA.$$

This is an example of a square matrix that has an inverse. Note that not all square matrices have inverses.

✓CHECKPOINT Now try Exercise 7.

Recall that it is not always true that $AB = BA$, even when both products are defined. However, if $A$ and $B$ are both square matrices and $AB = I_n$, then it can be shown that $BA = I_n$. So, in Example 1, you need only check that $AB = I_2$.

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## Finding Inverse Matrices

When a matrix $A$ has an inverse, $A$ is called **invertible** (or **nonsingular**); otherwise, $A$ is called **singular.** A nonsquare matrix cannot have an inverse. To see this, note that if $A$ is of dimension $m \times n$ and $B$ is of dimension $n \times m$ (where $m \neq n$), then the products $AB$ and $BA$ are of different dimensions and so cannot be equal to each other. Not all square matrices have inverses, as you will see later in this section. When a matrix does have an inverse, however, that inverse is unique. Example 2 shows how to use systems of equations to find the inverse of a matrix.

### Example 2 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

#### Solution

To find the inverse of $A$, try to solve the matrix equation

$$AX = I$$

for $X$.

$$\overset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, you obtain the following two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases}$$ Linear system with two variables, $x_{11}$ and $x_{21}$.

$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$ Linear system with two variables, $x_{12}$ and $x_{22}$.

Solve the first system using elementary row operations to determine that

$$x_{11} = -3 \quad \text{and} \quad x_{21} = 1.$$

From the second system you can determine that

$$x_{12} = -4 \quad \text{and} \quad x_{22} = 1.$$

Therefore, the inverse of $A$ is

$$A^{-1} = X$$
$$= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

You can use matrix multiplication to check this result.

#### Check

$$AA^{-1} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

**✓CHECKPOINT** Now try Exercise 11.

### Explore the Concept

Most graphing utilities are capable of finding the inverse of a square matrix. Try using a graphing utility to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 2 & -1 \\ -2 & 0 & 1 \end{bmatrix}.$$

After you find $A^{-1}$, store it as $[B]$ and use the graphing utility to find $[A] \times [B]$ and $[B] \times [A]$. What can you conclude?

In Example 2, note that the two systems of linear equations have the *same coefficient matrix* $A$. Rather than solve the two systems represented by

$$\left[\begin{array}{rrcr} 1 & 4 & \vdots & 1 \\ -1 & -3 & \vdots & 0 \end{array}\right]$$

and

$$\left[\begin{array}{rrcr} 1 & 4 & \vdots & 0 \\ -1 & -3 & \vdots & 1 \end{array}\right]$$

separately, you can solve them *simultaneously* by *adjoining* the identity matrix to the coefficient matrix to obtain

$$\begin{array}{c} \begin{array}{ccccc} & A & & I & \end{array} \\ \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{array}\right]. \end{array}$$

This "doubly augmented" matrix can be represented as

$$[A \ \vdots \ I].$$

By applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process.

$$\begin{array}{r} \\ \\ \\ R_1 + R_2 \rightarrow \\ -4R_2 + R_1 \rightarrow \\ \\ \end{array} \begin{array}{c} \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{array}\right] \\ \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 1 & 1 \end{array}\right] \\ \left[\begin{array}{rrcrr} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{array}\right] \end{array}$$

So, from the "doubly augmented" matrix $[A \ \vdots \ I]$, you obtained the matrix $[I \ \vdots \ A^{-1}]$.

$$\begin{array}{c} \begin{array}{ccccc} & A & & I & \end{array} \\ \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \begin{array}{ccccc} & I & & A^{-1} & \end{array} \\ \left[\begin{array}{rrcrr} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{array}\right] \end{array}$$

This procedure (or algorithm) works for any square matrix that has an inverse.

### Explore the Concept

Select two $2 \times 2$ matrices $A$ and $B$ that have inverses. Enter them into your graphing utility and calculate $(AB)^{-1}$. Then calculate $B^{-1}A^{-1}$ and $A^{-1}B^{-1}$. Make a conjecture about the inverse of the product of two invertible matrices.

### Finding an Inverse Matrix

Let $A$ be a square matrix of dimension $n \times n$.

**1.** Write the $n \times 2n$ matrix that consists of the given matrix $A$ on the left and the $n \times n$ identity matrix $I$ on the right to obtain

$$[A \ \vdots \ I].$$

**2.** If possible, row reduce $A$ to $I$ using elementary row operations on the *entire* matrix

$$[A \ \vdots \ I].$$

The result will be the matrix

$$[I \ \vdots \ A^{-1}].$$

If this is not possible, then $A$ is not invertible.

**3.** Check your work by multiplying to see that

$$AA^{-1} = I = A^{-1}A.$$

## Example 3 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}.$$

### Solution

Begin by adjoining the identity matrix to $A$ to form the matrix

$$[A \ \vdots \ I] = \left[\begin{array}{rrrcrrr} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ 6 & -2 & -3 & \vdots & 0 & 0 & 1 \end{array}\right].$$

Use elementary row operations to obtain the form $[I \ \vdots \ A^{-1}]$, as follows.

$$\left[\begin{array}{rrrcrrr} 1 & 0 & 0 & \vdots & -2 & -3 & 1 \\ 0 & 1 & 0 & \vdots & -3 & -3 & 1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & 1 \end{array}\right]$$

Therefore, the matrix $A$ is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}.$$

Confirm this result by multiplying $A$ by $A^{-1}$ to obtain $I$, as follows.

### Check

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

**✓CHECKPOINT** Now try Exercise 17.

## Technology Tip

Most graphing utilities can find the inverse of a matrix by using the inverse key $x^{-1}$. For instructions on how to use the inverse key to find the inverse of a matrix, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

The algorithm shown in Example 3 applies to any $n \times n$ matrix $A$. When using this algorithm, if the matrix $A$ does not reduce to the identity matrix, then $A$ does not have an inverse. For instance, the following matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

To see why matrix $A$ above has no inverse, begin by adjoining the identity matrix to $A$ to form

$$[A \ \vdots \ I] = \left[\begin{array}{rrrcrrr} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 3 & -1 & 2 & \vdots & 0 & 1 & 0 \\ -2 & 3 & -2 & \vdots & 0 & 0 & 1 \end{array}\right].$$

Then use elementary row operations to obtain

$$\left[\begin{array}{rrrcrrr} 1 & 2 & 0 & \vdots & 1 & 0 & 0 \\ 0 & -7 & 2 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 0 & \vdots & -1 & 1 & 1 \end{array}\right].$$

At this point in the elimination process you can see that it is impossible to obtain the identity matrix $I$ on the left. Therefore, $A$ is not invertible.

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## The Inverse of a 2 × 2 Matrix

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of dimension 3 × 3 or greater. For 2 × 2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2 × 2 matrices, is explained as follows. If $A$ is the 2 × 2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $A$ is invertible if and only if

$$ad - bc \neq 0.$$

If $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Formula for inverse of matrix $A$

The denominator

$$ad - bc$$

is called the *determinant* of the 2 × 2 matrix $A$. You will study determinants in the next section.

### Example 4 Finding the Inverse of a 2 × 2 Matrix

If possible, find the inverse of each matrix.

**a.** $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

**b.** $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

#### Solution

**a.** For the matrix $A$, apply the formula for the inverse of a 2 × 2 matrix to obtain

$$\begin{aligned} ad - bc &= (3)(2) - (-1)(-2) \\ &= 4. \end{aligned}$$

Because this quantity is not zero, the inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as follows.

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

Substitute for $a$, $b$, $c$, $d$, and the determinant.

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Multiply by the scalar $\frac{1}{4}$.

**b.** For the matrix $B$, you have

$$\begin{aligned} ad - bc &= (3)(2) - (-1)(-6) \\ &= 0 \end{aligned}$$

which means that $B$ is not invertible.

CHECKPOINT Now try Exercise 29.

### Explore the Concept

Use a graphing utility to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

What message appears on the screen? Why does the graphing utility display this message?

## Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix $A$ of a *square* system (a system that has the same number of equations as variables) is invertible, then the system has a unique solution, which is defined as follows.

### A System of Equations with a Unique Solution

If $A$ is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by

$$X = A^{-1}B.$$

The formula $X = A^{-1}B$ is used on most graphing utilities to solve linear systems that have invertible coefficient matrices. That is, you enter the $n \times n$ coefficient matrix $[A]$ and the $n \times 1$ column matrix $[B]$. The solution $X$ is given by $[A]^{-1}[B]$.

### Example 5 Solving a System of Equations Using an Inverse

Use an inverse matrix to solve the system.

$$\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = \phantom{-}1 \\ 2x + 4y + z = -2 \end{cases}$$

### Solution

Begin by writing the system as $AX = B$.

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

Then, use Gauss-Jordan elimination to find $A^{-1}$.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Finally, multiply $B$ by $A^{-1}$ on the left to obtain the solution.

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \end{aligned}$$

So, the solution is

$$x = 2, \quad y = -1, \quad \text{and} \quad z = -2.$$

Use a graphing utility to verify $A^{-1}$ for the system of equations.

**CHECKPOINT** Now try Exercise 51.

### Study Tip

Remember that matrix multiplication is not commutative. So, you must multiply matrices in the correct order. For instance, in Example 5, you must multiply $B$ by $A^{-1}$ on the left.

## 7.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1 and 2, fill in the blank(s).**

1. If there exists an $n \times n$ matrix $A^{-1}$ such that $AA^{-1} = I_n = A^{-1}A$, then $A^{-1}$ is called the ________ of $A$.
2. If a matrix $A$ has an inverse, then it is called invertible or ________ ; if it does not have an inverse, then it is called ________ .
3. Do all square matrices have inverses?
4. Given that $A$ and $B$ are square matrices and $AB = I_n$, does $BA = I_n$?

### Procedures and Problem Solving

**The Inverse of a Matrix** **In Exercises 5–10, show that $B$ is the inverse of $A$.**

**5.** $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

**6.** $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

✓ **7.** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

**8.** $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

**9.** $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

**10.** $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 1 \\ -3 & -2 & 1 \end{bmatrix}$

**Finding the Inverse of a Matrix** **In Exercises 11–20, find the inverse of the matrix (if it exists).**

✓ **11.** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

**12.** $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

**13.** $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

**14.** $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

**15.** $\begin{bmatrix} 2 & 7 & 1 \\ -3 & -9 & 2 \end{bmatrix}$

**16.** $\begin{bmatrix} -2 & 5 \\ 6 & -15 \\ 0 & 1 \end{bmatrix}$

✓ **17.** $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

**18.** $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

**19.** $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

**20.** $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 2 & 5 & 0 \end{bmatrix}$

**Finding the Inverse of a Matrix** **In Exercises 21–28, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).**

**21.** $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

**22.** $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

**23.** $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & -\frac{3}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

**24.** $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

**25.** $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

**26.** $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

**27.** $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

**28.** $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

**Finding the Inverse of a 2 × 2 Matrix** **In Exercises 29–34, use the formula on page 536 to find the inverse of the 2 × 2 matrix.**

✓ **29.** $\begin{bmatrix} 5 & 1 \\ -2 & -2 \end{bmatrix}$

**30.** $\begin{bmatrix} 7 & 12 \\ -8 & -5 \end{bmatrix}$

**31.** $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$

**32.** $\begin{bmatrix} -\frac{1}{4} & -\frac{2}{3} \\ \frac{1}{3} & \frac{8}{9} \end{bmatrix}$

**33.** $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$

**34.** $\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

**Finding a Matrix Entry** **In Exercises 35 and 36, find the value of the constant $k$ such that $B = A^{-1}$.**

**35.** $A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} k & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

**36.** $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ k & \frac{1}{3} \end{bmatrix}$

**Solving a System of Linear Equations** **In Exercises 37–40, use the inverse matrix found in Exercise 13 to solve the system of linear equations.**

**37.** $\begin{cases} x - 2y = 5 \\ 2x - 3y = 10 \end{cases}$ **38.** $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$

**39.** $\begin{cases} x - 2y = 4 \\ 2x - 3y = 2 \end{cases}$ **40.** $\begin{cases} x - 2y = 1 \\ 2x - 3y = -2 \end{cases}$

**Solving a System of Linear Equations** **In Exercises 41 and 42, use the inverse matrix found in Exercise 17 to solve the system of linear equations.**

**41.** $\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$ **42.** $\begin{cases} x + y + z = -1 \\ 3x + 5y + 4z = 2 \\ 3x + 6y + 5z = 0 \end{cases}$

**Solving a System of Linear Equations** **In Exercises 43 and 44, use the inverse matrix found in Exercise 28 and the matrix capabilities of a graphing utility to solve the system of linear equations.**

**43.** $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = 2 \end{cases}$

**44.** $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = -3 \end{cases}$

**Solving a System of Equations Using an Inverse** **In Exercises 45–52, use an inverse matrix to solve (if possible) the system of linear equations.**

**45.** $\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$ **46.** $\begin{cases} 18x + 12y = 13 \\ 30x + 24y = 23 \end{cases}$

**47.** $\begin{cases} -0.4x + 0.8y = 1.6 \\ 2x - 4y = 5 \end{cases}$ **48.** $\begin{cases} 0.2x - 0.6y = 2.4 \\ -x + 1.4y = -8.8 \end{cases}$

**49.** $\begin{cases} -\frac{1}{4}x + \frac{3}{8}y = -2 \\ \frac{3}{2}x + \frac{3}{4}y = -12 \end{cases}$ **50.** $\begin{cases} \frac{5}{6}x - y = -20 \\ \frac{4}{3}x - \frac{7}{2}y = -51 \end{cases}$

✓ **51.** $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$ **52.** $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

**Solving a System of Linear Equations** **In Exercises 53–56, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.**

**53.** $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ -x + 7y - 8z = 4 \end{cases}$ **54.** $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y - 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

**55.** $\begin{cases} 7x - 3y + 2w = 41 \\ -2x + y - w = -13 \\ 4x + z - 2w = 12 \\ -x + y - w = -8 \end{cases}$

**56.** $\begin{cases} 2x + 5y + w = 11 \\ x + 4y + 2z - 2w = -7 \\ 2x - 2y + 5z + w = 3 \\ x - 3w = -1 \end{cases}$

**Investment Portfolio** **In Exercises 57–60, consider a person who invests in AAA-rated bonds, A-rated bonds, and B-rated bonds. The average yields are 6.5% on AAA bonds, 7% on A bonds, and 9% on B bonds. The person invests twice as much in B bonds as in A bonds. Let $x$, $y$, and $z$ represent the amounts invested in AAA, A, and B bonds, respectively.**

$$\begin{cases} x + y + z = \text{(total investment)} \\ 0.065x + 0.07y + 0.09z = \text{(annual return)} \\ 2y - z = 0 \end{cases}$$

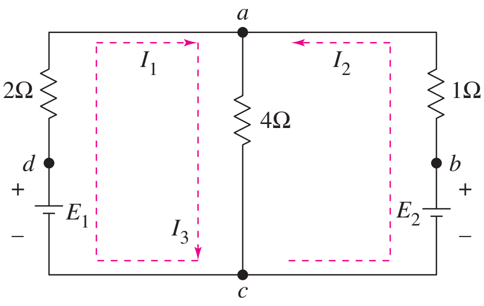
**Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond.**

| | *Total Investment* | *Annual Return* |
|---|---|---|
| **57.** | $10,000 | $705 |
| **58.** | $10,000 | $760 |
| **59.** | $12,000 | $835 |
| **60.** | $500,000 | $38,000 |

**Electrical Engineering** **In Exercises 61 and 62, consider the circuit in the figure. The currents $I_1$, $I_2$, and $I_3$, in amperes, are given by the solution of the system of linear equations**

$$\begin{cases} 2I_1 + 4I_3 = E_1 \\ I_2 + 4I_3 = E_2 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

**where $E_1$ and $E_2$ are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the given voltages.**



**61.** $E_1 = 15$ volts, $E_2 = 17$ volts

**62.** $E_1 = 10$ volts, $E_2 = 10$ volts

**Horticulture** **In Exercises 63 and 64, consider a company that specializes in potting soil. Each bag of potting soil for seedlings requires 2 units of sand, 1 unit of loam, and 1 unit of peat moss. Each bag of potting soil for general potting requires 1 unit of sand, 2 units of loam, and 1 unit of peat moss. Each bag of potting soil for hardwood plants requires 2 units of sand, 2 units of loam, and 2 units of peat moss. Find the numbers of bags of the three types of potting soil that the company can produce with the given amounts of raw materials.**

**63.** 500 units of sand
500 units of loam
400 units of peat moss

**64.** 500 units of sand
750 units of loam
450 units of peat moss

**65.** **Floral Design** A florist is creating 10 centerpieces for the tables at a wedding reception. Roses cost \$2.50 each, lilies cost \$4 each, and irises cost \$2 each. The customer has a budget of \$300 allocated for the centerpieces and wants each centerpiece to contain 12 flowers, with twice as many roses as the number of irises and lilies combined.

(a) Write a system of linear equations that represents the situation.

(b) Write a matrix equation that corresponds to your system.

(c) Solve your system of linear equations using an inverse matrix. Find the number of flowers of each type that the florist can use to create the 10 centerpieces.

**66.** ***Why you should learn it*** *(p. 532)* The table shows the numbers of international travelers $y$ (in thousands) to the United States from Europe from 2006 through 2008. (Source: U.S. Department of Commerce)

| Year | Travelers, $y$ (in thousands) |
|---|---|
| 2006 | 10,136 |
| 2007 | 11,406 |
| 2008 | 12,783 |

(a) The data can be approximated by a parabola. Create a system of linear equations for the data. Let $t$ represent the year, with $t = 6$ corresponding to 2006.

(b) Use the matrix capabilities of a graphing utility to find an inverse matrix to solve the system in part (a) and find the least squares regression parabola $y = at^2 + bt + c$.

(c) Use the graphing utility to graph the parabola with the data points.

(d) Use the result of part (b) to estimate the numbers of international travelers to the United States from Europe in 2009, 2010, and 2011.

(e) Are your estimates from part (d) reasonable? Explain.

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## Conclusions

**True or False?** **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

**67.** Multiplication of an invertible matrix and its inverse is commutative.

**68.** When the product of two square matrices is the identity matrix, the matrices are inverses of one another.

**69.** **Writing** Explain how to determine whether the inverse of a $2 \times 2$ matrix exists. If so, explain how to find the inverse.

**70.** If $A$ is a $2 \times 2$ matrix given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A$ is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, verify that the inverse is $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

**71.** **Exploration** Consider matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

(a) Write a $2 \times 2$ matrix and a $3 \times 3$ matrix in the form of $A$. Find the inverse of each.

(b) Use the result of part (a) to make a conjecture about the inverse of a matrix in the form of $A$.

**72.** **CAPSTONE** Let $A$ be a $2 \times 2$ matrix given by

$$A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}.$$

Use the determinant of $A$ to determine the conditions under which $A^{-1}$ exists.

## Cumulative Mixed Review

**Solving an Equation** **In Exercises 73–76, solve the equation algebraically. Round your result to three decimal places.**

**73.** $e^{2x} + 2e^x - 15 = 0$

**74.** $e^{2x} - 10e^x + 24 = 0$

**75.** $7 \ln 3x = 12$

**76.** $\ln(x + 9) = 2$

**77.** ***Make a Decision*** To work an extended application analyzing the numbers of U.S. households with televisions from 1985 through 2008, visit this textbook's *Companion Website.* (Data Source: Nielsen Media Research)

# 7.7 The Determinant of a Square Matrix

### *What you should learn*

- Find the determinants of 2 × 2 matrices.
- Find minors and cofactors of square matrices.
- Find the determinants of square matrices.

### *Why you should learn it*

Determinants and Cramer's Rule can be used to find the least squares regression parabola that models retail sales of family clothing stores, as shown in Exercise 27 on page 556 of Section 7.8.

## The Determinant of a $2 \times 2$ Matrix

Every *square* matrix can be associated with a real number called its **determinant.** Determinants have many uses, and several will be discussed in this and the next section. Historically, the use of determinants arose from special number patterns that occur when systems of linear equations are solved. For instance, the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that $a_1b_2 - a_2b_1 \neq 0$. Note that each fraction has the same denominator. This denominator is called the *determinant* of the coefficient matrix of the system.

*Coefficient Matrix* *Determinant*

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \qquad \det(A) = a_1b_2 - a_2b_1$$

The determinant of the matrix $A$ can also be denoted by vertical bars on both sides of the matrix, as indicated in the following definition.

### Definition of the Determinant of a $2 \times 2$ Matrix

The **determinant** of the matrix

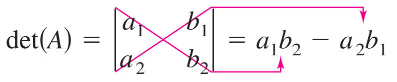
$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

is given by

$$\begin{aligned} \det(A) &= |A| \\ &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= a_1b_2 - a_2b_1. \end{aligned}$$

In this text, $\det(A)$ and $|A|$ are used interchangeably to represent the determinant of $A$. Although vertical bars are also used to denote the absolute value of a real number, the context will show which use is intended.

A convenient method for remembering the formula for the determinant of a $2 \times 2$ matrix is shown in the following diagram.



$$\det(A) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Note that the determinant is the difference of the products of the two diagonals of the matrix.

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### Example 1 The Determinant of a 2 × 2 Matrix

Find the determinant of each matrix.

**a.** $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

**b.** $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

**c.** $C = \begin{bmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{bmatrix}$

### Solution

**a.** $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3)$

$= 4 + 3$

$= 7$

**b.** $\det(B) = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1)$

$= 4 - 4$

$= 0$

**c.** $\det(C) = \begin{vmatrix} 0 & \frac{3}{2} \\ 2 & 4 \end{vmatrix} = 0(4) - 2\left(\frac{3}{2}\right)$

$= 0 - 3$

$= -3$

**CHECKPOINT** Now try Exercise 9.

Notice in Example 1 that the determinant of a matrix can be positive, zero, or negative.

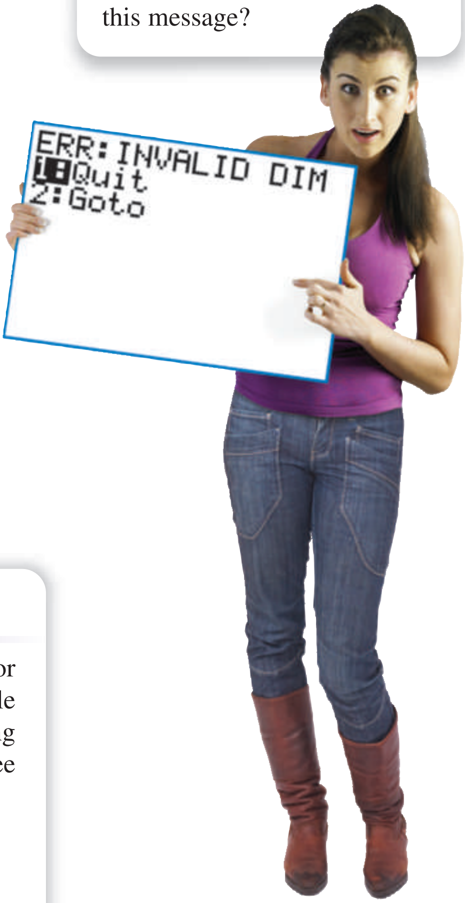
The determinant of a matrix of dimension 1 × 1 is defined simply as the entry of the matrix. For instance, if $A = [-2]$, then $\det(A) = -2$.

## Explore the Concept

Try using a graphing utility to find the determinant of

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

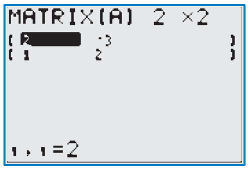
What message appears on the screen? Why does the graphing utility display this message?



```
ERR:INVALID DIM
1:Quit
2:Goto
```

## Technology Tip

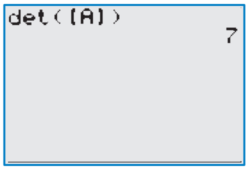
Most graphing utilities can evaluate the determinant of a matrix. For instance, you can evaluate the determinant of the matrix $A$ in Example 1(a) by entering the matrix as $[A]$ (see Figure 7.26) and then choosing the *determinant* feature. The result should be 7, as in Example 1(a) (see Figure 7.27).



```
MATRIX[A]  2 ×2
[2    -3      ]
[1     2      ]

1,1=2
```

**Figure 7.26**



```
det([A])
                  7
```

**Figure 7.27**

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Vibrant Image Studio 2010/used under license from Shutterstock.com

## Minors and Cofactors

To define the determinant of a square matrix of dimension 3 × 3 or higher, it is helpful to introduce the concepts of **minors** and **cofactors.**

### Minors and Cofactors of a Square Matrix

If $A$ is a square matrix, then the **minor** $M_{ij}$ of the entry $a_{ij}$ is the determinant of the matrix obtained by deleting the $i$th row and $j$th column of $A$. The **cofactor** $C_{ij}$ of the entry $a_{ij}$ is given by

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

In the sign patterns for cofactors at the right, notice that *odd* positions (where $i + j$ is odd) have negative signs and *even* positions (where $i + j$ is even) have positive signs.

*Sign Patterns for Cofactors*

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3 × 3 matrix

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4 × 4 matrix

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

### Example 2 Finding the Minors and Cofactors of a Matrix

Find all the minors and cofactors of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

#### Solution

To find the minor

$$M_{11}$$

delete the first row and first column of $A$ and evaluate the determinant of the resulting matrix.

$$\begin{bmatrix} \boxed{0} & \cancel{2} & \cancel{1} \\ \cancel{3} & -1 & 2 \\ \cancel{4} & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 0(2) = -1$$

Similarly, to find the minor

$$M_{12}$$

delete the first row and second column.

$$\begin{bmatrix} \cancel{0} & \boxed{2} & \cancel{1} \\ 3 & \cancel{-1} & 2 \\ 4 & \cancel{0} & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

Continuing this pattern, you obtain all the minors.

$$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = \phantom{-}4 \\ M_{21} = \phantom{-}2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = \phantom{-}5 & M_{32} = -3 & M_{33} = -6 \end{array}$$

Now, to find the cofactors, combine these minors with the checkerboard pattern of signs for a 3 × 3 matrix shown at the upper right.

$$\begin{array}{lll} C_{11} = -1 & C_{12} = \phantom{-}5 & C_{13} = \phantom{-}4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = \phantom{-}8 \\ C_{31} = \phantom{-}5 & C_{32} = \phantom{-}3 & C_{33} = -6 \end{array}$$

CHECKPOINT Now try Exercise 17.

## The Determinant of a Square Matrix

The following definition is called *inductive* because it uses determinants of matrices of dimension $(n - 1) \times (n - 1)$ to define determinants of matrices of dimension $n \times n$.

### Determinant of a Square Matrix

If $A$ is a square matrix (of dimension $2 \times 2$ or greater), then the determinant of $A$ is the sum of the entries in any row (or column) of $A$ multiplied by their respective cofactors. For instance, expanding along the first row yields

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdot \cdot \cdot + a_{1n}C_{1n}.$$

Applying this definition to find a determinant is called **expanding by cofactors.**

Try checking that for a $2 \times 2$ matrix

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

this definition of the determinant yields $|A| = a_1b_2 - a_2b_1$, as previously defined.

## Example 3 The Determinant of a Matrix of Dimension 3 × 3

Find the determinant of $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$.

### Solution

Note that this is the same matrix that was in Example 2. There you found the cofactors of the entries in the first row to be

$$C_{11} = -1, \quad C_{12} = 5, \quad \text{and} \quad C_{13} = 4.$$

So, by the definition of the determinant of a square matrix, you have

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{First-row expansion} \\ &= 0(-1) + 2(5) + 1(4) \\ &= 14. \end{aligned}$$

**CHECKPOINT** Now try Exercise 23.

In Example 3, the determinant was found by expanding by the cofactors in the first row. You could have used any row or column. For instance, you could have expanded along the second row to obtain

$$\begin{aligned} |A| &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{Second-row expansion} \\ &= 3(-2) + (-1)(-4) + 2(8) \\ &= 14. \end{aligned}$$

When expanding by cofactors, you do not need to find cofactors of zero entries, because zero times its cofactor is zero.

$$a_{ij}C_{ij} = (0)C_{ij} = 0$$

So, the row (or column) containing the most zeros is usually the best choice for expansion by cofactors.

## 7.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1 and 2, fill in the blank.**

**1.** Both det($A$) and $|A|$ represent the _______ of the matrix $A$.

**2.** The determinant of the matrix obtained by deleting the $i$th row and $j$th column of a square matrix $A$ is called the _______ of the entry $a_{ij}$.

**3.** For a square matrix $B$, the minor $M_{23} = 5$. What is the cofactor $C_{23}$ of matrix $B$?

**4.** To find the determinant of a matrix using expanding by cofactors, do you need to find all the cofactors?

### Procedures and Problem Solving

**The Determinant of a Matrix** **In Exercises 5–12, find the determinant of the matrix.**

**5.** $[4]$

**6.** $[-12]$

**7.** $\begin{bmatrix} 8 & 4 \\ 2 & 3 \end{bmatrix}$

**8.** $\begin{bmatrix} -5 & 2 \\ 6 & 3 \end{bmatrix}$

✓ **9.** $\begin{bmatrix} 6 & 2 \\ -5 & 3 \end{bmatrix}$

**10.** $\begin{bmatrix} 3 & -3 \\ 4 & -8 \end{bmatrix}$

**11.** $\begin{bmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{bmatrix}$

**12.** $\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$

**Using a Graphing Utility** **In Exercises 13 and 14, use the matrix capabilities of a graphing utility to find the determinant of the matrix.**

**13.** $\begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 \\ -0.4 & 0.4 & 0.3 \end{bmatrix}$

**14.** $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

**Finding the Minors and Cofactors of a Matrix** **In Exercises 15–18, find all (a) minors and (b) cofactors of the matrix.**

**15.** $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$

**16.** $\begin{bmatrix} 11 & 0 \\ -3 & 2 \end{bmatrix}$

✓ **17.** $\begin{bmatrix} -4 & 6 & 3 \\ 7 & -2 & 8 \\ 1 & 0 & -5 \end{bmatrix}$

**18.** $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

**Finding a Determinant** **In Exercises 19–22, find the determinant of the matrix. Expand by cofactors on each indicated row or column.**

**19.** $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$

(a) Row 1

(b) Column 2

**20.** $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

(a) Row 2

(b) Column 3

**21.** $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 13 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 6 & 0 & 2 \end{bmatrix}$

(a) Row 2

(b) Column 2

**22.** $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 1 & 0 & -3 & 2 \end{bmatrix}$

(a) Row 3

(b) Column 1

**Finding a Determinant** **In Exercises 23–32, find the determinant of the matrix. Expand by cofactors on the row or column that appears to make the computations easiest.**

✓ **23.** $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

**24.** $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

**25.** $\begin{bmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{bmatrix}$

**26.** $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

**27.** $\begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix}$

**28.** $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 4 & 1 & 5 \end{bmatrix}$

**29.** $\begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 5 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$

**30.** $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$

**31.** $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

**32.** $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

**Using a Graphing Utility** **In Exercises 33–36, use the matrix capabilities of a graphing utility to evaluate the determinant.**

**33.** $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix}$ **34.** $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

**35.** $\begin{vmatrix} 3 & -2 & 4 & 3 & 1 \\ -1 & 0 & 2 & 1 & 0 \\ 5 & -1 & 0 & 3 & 2 \\ 4 & 7 & -8 & 0 & 0 \\ 1 & 2 & 3 & 0 & 2 \end{vmatrix}$

**36.** $\begin{vmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{vmatrix}$

**The Determinant of a Matrix Product** **In Exercises 37–40, find (a) $|A|$, (b) $|B|$, (c) $AB$, and (d) $|AB|$. What do you notice about $|AB|$?**

**37.** $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

**38.** $A = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

**39.** $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**40.** $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

**Using a Graphing Utility** **In Exercises 41 and 42, use the matrix capabilities of a graphing utility to find (a) $|A|$, (b) $|B|$, (c) $AB$, and (d) $|AB|$. What do you notice about $|AB|$?**

**41.** $A = \begin{bmatrix} 6 & 4 & 0 & 1 \\ 2 & -3 & -2 & -4 \\ 0 & 1 & 5 & 0 \\ -1 & 0 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -5 & 0 & -2 \\ -2 & 4 & -1 & -4 \\ 3 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 \end{bmatrix}$

**42.** $A = \begin{bmatrix} -1 & 5 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & -3 & -1 & 0 \\ 4 & 2 & 4 & -1 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 10 & -1 & 2 & 4 \\ 2 & 0 & 0 & 1 \\ -3 & 2 & 5 & 0 \end{bmatrix}$

**Verifying an Equation** **In Exercises 43–48, evaluate the determinants to verify the equation.**

**43.** $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$

**44.** $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

**45.** $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$

**46.** $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

**47.** $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - x)(z - x)(z - y)$

**48.** $\begin{vmatrix} a + b & a & a \\ a & a + b & a \\ a & a & a + b \end{vmatrix} = b^2(3a + b)$

**Solving an Equation** **In Exercises 49–60, solve for $x$.**

**49.** $\begin{vmatrix} x & 2 \\ 1 & x \end{vmatrix} = 2$ **50.** $\begin{vmatrix} x & 4 \\ -1 & x \end{vmatrix} = 20$

**51.** $\begin{vmatrix} 2x & -3 \\ -2 & 2x \end{vmatrix} = 3$ **52.** $\begin{vmatrix} x & 2 \\ 4 & 9x \end{vmatrix} = 8$

**53.** $\begin{vmatrix} x & 1 \\ 2 & x - 2 \end{vmatrix} = -1$ **54.** $\begin{vmatrix} x + 1 & 2 \\ -1 & x \end{vmatrix} = 4$

**55.** $\begin{vmatrix} x + 3 & 2 \\ 1 & x + 2 \end{vmatrix} = 0$ **56.** $\begin{vmatrix} x - 2 & -1 \\ -3 & x \end{vmatrix} = 0$

**57.** $\begin{vmatrix} 2x & 1 \\ -1 & x - 1 \end{vmatrix} = x$ **58.** $\begin{vmatrix} x - 1 & x \\ x + 1 & 2 \end{vmatrix} = -8$

**59.** $\begin{vmatrix} 1 & 2 & x \\ -1 & 3 & 2 \\ 3 & -2 & 1 \end{vmatrix} = 0$ **60.** $\begin{vmatrix} 1 & x & -2 \\ 1 & 3 & 3 \\ 0 & 2 & -2 \end{vmatrix} = 0$

∫ **Entries Involving Expressions** **In Exercises 61–66, evaluate the determinant, in which the entries are functions. Determinants of this type occur when changes of variables are made in calculus.**

**61.** $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix}$

**62.** $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$

**63.** $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

**64.** $\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1 - x)e^{-x} \end{vmatrix}$

**65.** $\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$

**66.** $\begin{vmatrix} x & x \ln x \\ 1 & 1 + \ln x \end{vmatrix}$

## Conclusions

**True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

**67.** If a square matrix has an entire row of zeros, then the determinant will always be zero.

**68.** If two columns of a square matrix are the same, then the determinant of the matrix will be zero.

**69. Exploration** Find a pair of $3 \times 3$ matrices $A$ and $B$ to demonstrate that $|A + B| \neq |A| + |B|$.

**70. Think About It** Let $A$ be a $3 \times 3$ matrix such that $|A| = 5$. Can you use this information to find $|2A|$? Explain.

**Exploration In Exercises 71–74, (a) find the determinant of $A$, (b) find $A^{-1}$, (c) find $\det(A^{-1})$, and (d) compare your results from parts (a) and (c). Make a conjecture based on your results.**

**71.** $A = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$ **72.** $A = \begin{bmatrix} 5 & -1 \\ 2 & -1 \end{bmatrix}$

**73.** $A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 3 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ **74.** $A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

**Properties of Determinants In Exercises 75–77, a property of determinants is given ($A$ and $B$ are square matrices). State how the property has been applied to the given determinants and use a graphing utility to verify the results.**

**75.** If $B$ is obtained from $A$ by interchanging two rows of $A$ or by interchanging two columns of $A$, then $|B| = -|A|$.

(a) $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

**76.** If $B$ is obtained from $A$ by adding a multiple of a row of $A$ to another row of $A$ or by adding a multiple of a column of $A$ to another column of $A$, then $|B| = |A|$.

(a) $\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 0 & 17 \end{vmatrix}$

(b) $\begin{vmatrix} 5 & 4 & 2 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 10 & -6 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix}$

**77.** If $B$ is obtained from $A$ by multiplying a row of $A$ by a nonzero constant $c$ or by multiplying a column of $A$ by a nonzero constant $c$, then $|B| = c|A|$.

(a) $\begin{vmatrix} 1 & 5 \\ 6 & 9 \end{vmatrix} = 3\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 8 \\ 6 & 8 \end{vmatrix} = 8\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$

**78. Exploration** A **diagonal matrix** is a square matrix with all zero entries above and below its main diagonal. Evaluate the determinant of each diagonal matrix. Make a conjecture based on your results.

(a) $\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

**79. Exploration** A **triangular matrix** is a square matrix with all zero entries either above or below its main diagonal. Such a matrix is **upper triangular** when it has all zeros below the main diagonal and **lower triangular** when it has all zeros above the main diagonal. A diagonal matrix is both upper and lower triangular. Evaluate the determinant of each triangular matrix. Make a conjecture based on your results.

(a) $\begin{bmatrix} 3 & -2 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -7 & 1 \\ 0 & -5 & -9 \\ 0 & 0 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 3 & 6 & 5 & 0 \\ 2 & -2 & 1 & 2 \end{bmatrix}$

**80. CAPSTONE** Create a study sheet showing the methods you have learned for finding the determinant of a square matrix.

**81. Proof** Use your results in Exercises 37–42 to make a conjecture about the value of $|AB|$ for two $m \times m$ matrices $A$ and $B$. Prove your conjecture for $2 \times 2$ matrices.

**82. Exploration** Consider square matrices in which the entries are consecutive integers. An example of such a matrix is

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

Use a graphing utility to evaluate four determinants of this type. Make a conjecture based on the results. Then verify your conjecture.

## Cumulative Mixed Review

**Factoring a Quadratic Expression In Exercises 83–86, factor the expression.**

**83.** $x^2 - 3x + 2$ **84.** $x^2 + 5x + 6$

**85.** $4y^2 - 12y + 9$ **86.** $4y^2 - 28y + 49$

**Solving a System of Equations In Exercises 87 and 88, solve the system of equations using the method of substitution or the method of elimination.**

**87.** $\begin{cases} 3x - 10y = 46 \\ x + y = -2 \end{cases}$ **88.** $\begin{cases} 5x + 7y = 23 \\ -4x - 2y = -4 \end{cases}$

# 7.8 Applications of Matrices and Determinants

## Area of a Triangle

In this section, you will study some additional applications of matrices and determinants. The first involves a formula for finding the area of a triangle whose vertices are given by three points on a rectangular coordinate system.

### Area of a Triangle

The area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ is

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the symbol $(\pm)$ indicates that the appropriate sign should be chosen to yield a positive area.

### What you should learn

- Use determinants to find areas of triangles.
- Use determinants to decide whether points are collinear.
- Use Cramer's Rule to solve systems of linear equations.
- Use matrices to encode and decode messages.

### Why you should learn it

Matrices can be used to decode a message, as shown in Exercise 36 on page 557.

### Example 1 Finding the Area of a Triangle

Find the area of the triangle whose vertices are

$$(1, 0), \quad (2, 2), \quad \text{and} \quad (4, 3)$$

as shown in Figure 7.28.

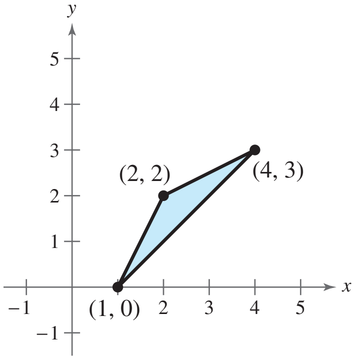

Figure 7.28

### Solution

Begin by letting

$$(x_1, y_1) = (1, 0), \quad (x_2, y_2) = (2, 2),$$

and

$$(x_3, y_3) = (4, 3).$$

Then, to find the area of the triangle, evaluate the determinant by expanding along row 1.

$$\begin{aligned}
\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} \\
&= 1(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 0(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} \\
&= 1(-1) + 0 + 1(-2) \\
&= -3
\end{aligned}$$

Using this value, you can conclude that the area of the triangle is

$$\begin{aligned}
\text{Area} &= -\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} \\
&= -\frac{1}{2}(-3) \\
&= \frac{3}{2} \text{ square units.}
\end{aligned}$$

✓CHECKPOINT Now try Exercise 5.

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## Collinear Points

What if the three points in Example 1 had been on the same line? What would have happened had the area formula been applied to three such points? The answer is that the determinant would have been zero. Consider, for instance, the three collinear points (0, 1), (2, 2), and (4, 3), as shown in Figure 7.29. The area of the "triangle" that has these three points as vertices is

$$\frac{1}{2}\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[0(-1)^2\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 1(-1)^3\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}\right]$$

$$= \frac{1}{2}[0 - 1(-2) + 1(-2)]$$

$$= 0$$

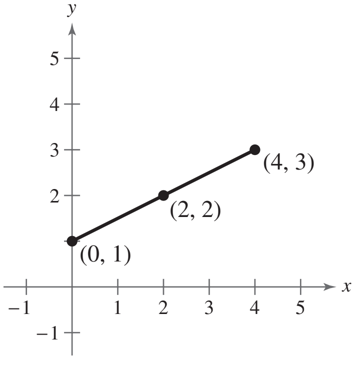


Figure 7.29

This result is generalized as follows.

### Test for Collinear Points

Three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ are **collinear** (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

### Example 2 Testing for Collinear Points

Determine whether the points

$(-2, -2)$, $(1, 1)$, and $(7, 5)$

are collinear. (See Figure 7.30.)

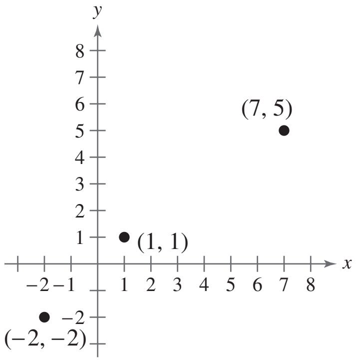


Figure 7.30

#### Solution

Begin by letting

$$(x_1, y_1) = (-2, -2), \quad (x_2, y_2) = (1, 1),$$

and

$$(x_3, y_3) = (7, 5).$$

Then by expanding along row 1, you have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 1 & 1 \\ 7 & 5 & 1 \end{vmatrix}$$

$$= -2(-1)^2\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + (-2)(-1)^3\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix} + 1(-1)^4\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix}$$

$$= -2(-4) + 2(-6) + 1(-2)$$

$$= -6.$$

Because the value of this determinant is *not* zero, you can conclude that the three points are not collinear.

CHECKPOINT Now try Exercise 13.

## Cramer's Rule

So far, you have studied three methods for solving a system of linear equations: substitution, elimination with equations, and elimination with matrices. You will now study one more method, **Cramer's Rule,** named after Gabriel Cramer (1704–1752). This rule uses determinants to write the solution of a system of linear equations. To see how Cramer's Rule works, take another look at the solution described at the beginning of Section 7.7. There, it was pointed out that the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

and

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that

$$a_1b_2 - a_2b_1 \neq 0.$$

Each numerator and denominator in this solution can be expressed as a determinant, as follows.

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Relative to the original system, the denominators of $x$ and $y$ are simply the determinant of the *coefficient* matrix of the system. This determinant is denoted by $D$. The numerators of $x$ and $y$ are denoted by $D_x$ and $D_y$, respectively. They are formed by using the column of constants as replacements for the coefficients of $x$ and $y$, as follows.

| *Coefficient Matrix* | $D$ | $D_x$ | $D_y$ |
|---|---|---|---|
| $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ | $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ | $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ | $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ |

For example, given the system

$$\begin{cases} 2x - 5y = 3 \\ -4x + 3y = 8 \end{cases}$$

the coefficient matrix, $D$, $D_x$, and $D_y$ are as follows.

| *Coefficient Matrix* | $D$ | $D_x$ | $D_y$ |
|---|---|---|---|
| $\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$ | $\begin{vmatrix} 2 & -5 \\ -4 & 3 \end{vmatrix}$ | $\begin{vmatrix} 3 & -5 \\ 8 & 3 \end{vmatrix}$ | $\begin{vmatrix} 2 & 3 \\ -4 & 8 \end{vmatrix}$ |

Cramer's Rule generalizes easily to systems of $n$ equations in $n$ variables. The value of each variable is given as the quotient of two determinants. The denominator is the determinant of the coefficient matrix, and the numerator is the determinant of the matrix formed by replacing the column corresponding to the variable being solved for with the column representing the constants. For instance, the solution for $x_3$ in the following system is shown.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \qquad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

### Cramer's Rule

If a system of $n$ linear equations in $n$ variables has a coefficient matrix $A$ with a *nonzero* determinant $|A|$, then the solution of the system is

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad . \ . \ . \ , \quad x_n = \frac{|A_n|}{|A|}$$

where the $i$th column of $A_i$ is the column of constants in the system of equations. If the determinant of the coefficient matrix is zero, then the system has either no solution or infinitely many solutions.

## Example 3 Using Cramer's Rule for a 2 × 2 System

Use Cramer's Rule to solve the system

$$\begin{cases} 4x - 2y = 10 \\ 3x - 5y = 11 \end{cases}.$$

### Solution

To begin, find the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = -20 - (-6) = -14$$

Because this determinant is not zero, apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix}}{-14} = \frac{(-50) - (-22)}{-14} = \frac{-28}{-14} = 2$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{-14} = \frac{44 - 30}{-14} = \frac{14}{-14} = -1$$

So, the solution is

$$x = 2 \quad \text{and} \quad y = -1.$$

Check this in the original system.

✓CHECKPOINT Now try Exercise 19.

### Example 4 Using Cramer's Rule for a 3 × 3 System

Use Cramer's Rule and a graphing utility, if possible, to solve the system of linear equations.

$$\begin{cases} -x \phantom{{}-y} + z = 4 \\ 2x - y + z = -3 \\ \phantom{2x -{}} y - 3z = 1 \end{cases}$$

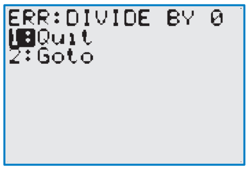
### Solution

Using a graphing utility to evaluate the determinant of the coefficient matrix $A$, you find that Cramer's Rule cannot be applied because $|A| = 0$.

**CHECKPOINT** Now try Exercise 21.

## Technology Tip

Try using a graphing utility to evaluate $D_x/D$ from Example 4. You should obtain the error message shown below.



```
ERR:DIVIDE BY 0
1:Quit
2:Goto
```

### Example 5 Using Cramer's Rule for a 3 × 3 System

Use Cramer's Rule, if possible, to solve the system of linear equations.

*Coefficient Matrix*

$$\begin{cases} -x + 2y - 3z = 1 \\ 2x \phantom{{}+2y} + z = 0 \\ 3x - 4y + 4z = 2 \end{cases} \Rightarrow \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix}$$

### Solution

The coefficient matrix above can be expanded along the second row, as follows.

$$\begin{aligned} D &= 2(-1)^3\begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix} + 0(-1)^4\begin{vmatrix} -1 & -3 \\ 3 & 4 \end{vmatrix} + 1(-1)^5\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} \\ &= -2(-4) + 0 - 1(-2) \\ &= 10 \end{aligned}$$

Because this determinant is not zero, you can apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{8}{10} = \frac{4}{5}$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}}{10} = \frac{-15}{10} = -\frac{3}{2}$$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix}}{10} = \frac{-16}{10} = -\frac{8}{5}$$

The solution is

$\left(\frac{4}{5}, -\frac{3}{2}, -\frac{8}{5}\right)$.

Check this in the original system.

**CHECKPOINT** Now try Exercise 23.

Remember that Cramer's Rule does not apply when the determinant of the coefficient matrix is zero. This would create division by zero, which is undefined.

## Cryptography

A **cryptogram** is a message written according to a secret code. (The Greek word *kryptos* means "hidden.") Matrix multiplication can be used to encode and decode messages. To begin, you need to assign a number to each letter in the alphabet (with 0 assigned to a blank space), as follows.

0 = _
1 = A
2 = B
3 = C
4 = D
5 = E
6 = F
7 = G
8 = H

9 = I
10 = J
11 = K
12 = L
13 = M
14 = N
15 = O
16 = P
17 = Q

18 = R
19 = S
20 = T
21 = U
22 = V
23 = W
24 = X
25 = Y
26 = Z

**Applied Cryptography Researcher**

Then the message is converted to numbers and partitioned into **uncoded row matrices,** each having $n$ entries, as demonstrated in Example 6.

### Example 6 Forming Uncoded Row Matrices

Write the uncoded row matrices of dimension 1 × 3 for the message

MEET ME MONDAY.

### Solution

Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the following uncoded row matrices.

$$[13 \quad 5 \quad 5] \quad [20 \quad 0 \quad 13] \quad [5 \quad 0 \quad 13] \quad [15 \quad 14 \quad 4] \quad [1 \quad 25 \quad 0]$$

M E E T M E M O N D A Y

Note that a blank space is used to fill out the last uncoded row matrix.

CHECKPOINT Now try Exercise 29(a).

To encode a message, use the techniques demonstrated in Section 7.6 to choose an $n \times n$ invertible matrix such as

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

and multiply the uncoded row matrices by $A$ (on the right) to obtain **coded row matrices.** Here is an example.

*Uncoded Matrix* *Encoding Matrix A* *Coded Matrix*

$$[13 \quad 5 \quad 5] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

This technique is further illustrated in Example 7.

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## Example 7 Encoding a Message

Use the matrix $A$ to encode the message

MEET ME MONDAY.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

### Solution

The coded row matrices are obtained by multiplying each of the uncoded row matrices found in Example 6 by the matrix $A$, as follows.

*Uncoded Matrix* *Encoding Matrix A* *Coded Matrix*

$$[13 \quad 5 \quad 5]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \quad -26 \quad 21]$$

$$[20 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [33 \quad -53 \quad -12]$$

$$[5 \quad 0 \quad 13]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [18 \quad -23 \quad -42]$$

$$[15 \quad 14 \quad 4]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [5 \quad -20 \quad 56]$$

$$[1 \quad 25 \quad 0]\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [-24 \quad 23 \quad 77]$$

So, the sequence of coded row matrices is

$$[13 \ -26 \ 21]\,[33 \ -53 \ -12]\,[18 \ -23 \ -42]\,[5 \ -20 \ 56]\,[-24 \ 23 \ 77].$$

Finally, removing the matrix notation produces the following cryptogram.

$$13 \ -26 \ 21 \ 33 \ -53 \ -12 \ 18 \ -23 \ -42 \ 5 \ -20 \ 56 \ -24 \ 23 \ 77$$

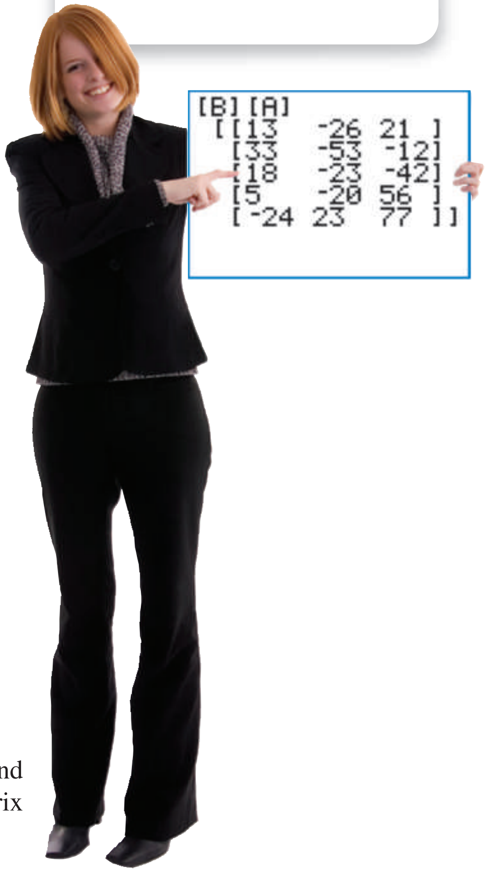
**CHECKPOINT** Now try Exercise 29(b).

## Technology Tip

An efficient method for encoding the message at the left with your graphing utility is to enter $A$ as a $3 \times 3$ matrix. Let $B$ be the $5 \times 3$ matrix whose rows are the uncoded row matrices

$$B = \begin{bmatrix} 13 & 5 & 5 \\ 20 & 0 & 13 \\ 5 & 0 & 13 \\ 15 & 14 & 4 \\ 1 & 25 & 0 \end{bmatrix}.$$

The product $BA$ gives the coded row matrices.



```
[B][A]
 [[13   -26  21 ]
  [33   -53  -12]
  [18   -23  -42]
  [5    -20  56 ]
  [-24  23   77 ]]
```

For those who do not know the encoding matrix $A$, decoding the cryptogram found in Example 7 is difficult. But for an authorized receiver who knows the encoding matrix $A$, decoding is simple. The receiver need only multiply the coded row matrices by

$$A^{-1}$$

(on the right) to retrieve the uncoded row matrices. Here is an example.

$$\underbrace{[13 \quad -26 \quad 21]}_{\text{Coded}}\underbrace{\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}}_{A^{-1}} = \underbrace{[13 \quad 5 \quad 5]}_{\text{Uncoded}}$$

This technique is further illustrated in Example 8.

Nathan Maxfield/iStockphoto.com

### Example 8 Decoding a Message

Use the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

to decode the cryptogram

13 −26 21 33 −53 −12 18 −23 −42 5 −20 56 −24 23 77.

#### Solution

First, find the decoding matrix $A^{-1}$ by using the techniques demonstrated in Section 7.6. Next partition the message into groups of three to form the coded row matrices. Then multiply each coded row matrix by $A^{-1}$ (on the right).

*Coded Matrix* *Decoding Matrix* $A^{-1}$ *Decoded Matrix*

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 14 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

So, the message is as follows.

$$\begin{array}{ccccccccccccccc} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ \text{M} & \text{E} & \text{E} & \text{T} & & \text{M} & \text{E} & & \text{M} & \text{O} & \text{N} & \text{D} & \text{A} & \text{Y} & \end{array}$$

✓CHECKPOINT Now try Exercise 35.

### Technology Tip

An efficient method for decoding the cryptogram in Example 8 with your graphing utility is to enter $A$ as a 3 × 3 matrix and then find $A^{-1}$. Let $B$ be the 5 × 3 matrix whose rows are the coded row matrices, as shown below. The product $BA^{-1}$ gives the decoded row matrices.

$$B = \begin{bmatrix} 13 & -26 & 21 \\ 33 & -53 & -12 \\ 18 & -23 & -42 \\ 5 & -20 & 56 \\ -24 & 23 & 77 \end{bmatrix}$$

## 7.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

### Vocabulary and Concept Check

**In Exercises 1 and 2, fill in the blank.**

**1.** ________ is a method for using determinants to solve a system of linear equations.

**2.** A message written according to a secret code is called a ________ .

**In Exercises 3 and 4, consider three points $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$, and the determinant shown at the right.**

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

**3.** Suppose the three points are vertices of a triangle and the value of the determinant is $-6$. What number do you multiply $-6$ by to find the area of the triangle?

**4.** Suppose the value of the determinant is 0. What can you conclude?

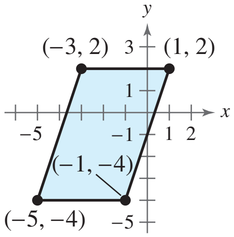
### Procedures and Problem Solving

**Finding an Area** **In Exercises 5–10, use a determinant to find the area of the figure with the given vertices.**

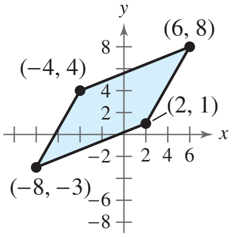
✓ **5.** $(-2, 4), (2, 3), (-1, 5)$ **6.** $(-3, 5), (2, 6), (3, -5)$

**7.** $\left(0, \frac{1}{2}\right), \left(\frac{5}{2}, 0\right), (4, 3)$ **8.** $\left(\frac{9}{2}, 0\right), (2, 6), \left(0, -\frac{3}{2}\right)$

**9.**



**10.**



**Finding a Coordinate** **In Exercises 11 and 12, find $x$ or $y$ such that the triangle has an area of 4 square units.**

**11.** $(-1, 5), (-2, 0), (x, 2)$ **12.** $(-4, 2), (-3, 5), (-1, y)$

**Testing for Collinear Points** **In Exercises 13–16, use a determinant to determine whether the points are collinear.**

✓ **13.** $(3, -1), (0, -3), (12, 5)$ **14.** $(3, -5), (6, 1), (4, 2)$

**15.** $\left(2, -\frac{1}{2}\right), (-4, 4), (6, -3)$ **16.** $\left(0, \frac{1}{2}\right), (2, -1), \left(-4, \frac{7}{2}\right)$

**Finding a Coordinate** **In Exercises 17 and 18, find $x$ or $y$ such that the points are collinear.**

**17.** $(1, -2), (x, 2), (5, 6)$ **18.** $(-6, 2), (-5, y), (-3, 5)$

**Using Cramer's Rule** **In Exercises 19–24, use Cramer's Rule to solve (if possible) the system of equations.**

✓ **19.** $\begin{cases} -7x + 11y = -1 \\ 3x - 9y = 9 \end{cases}$ **20.** $\begin{cases} 4x - 3y = -10 \\ 6x + 9y = 12 \end{cases}$

✓ **21.** $\begin{cases} 3x + 2y = -2 \\ 6x + 4y = 4 \end{cases}$ **22.** $\begin{cases} 6x - 5y = 17 \\ -13x + 3y = -76 \end{cases}$

✓ **23.** $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$ **24.** $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

**Comparing Solution Methods** **In Exercises 25 and 26, solve the system of equations using (a) Gaussian elimination and (b) Cramer's Rule. Which method do you prefer, and why?**

**25.** $\begin{cases} 3x + 3y + 5z = 1 \\ 3x + 5y + 9z = 2 \\ 5x + 9y + 17z = 4 \end{cases}$ **26.** $\begin{cases} 2x + 3y - 5z = 1 \\ 3x + 5y + 9z = -16 \\ 5x + 9y + 17z = -30 \end{cases}$

**27.** ***Why you should learn it*** *(p. 541)* The retail sales (in billions of dollars) of family clothing stores in the United States from 2004 through 2008 are shown in the table. (Source: U.S. Census Bureau)

| Year | Sales (in billions of dollars) |
|---|---|
| 2004 | 72.0 |
| 2005 | 77.4 |
| 2006 | 82.0 |
| 2007 | 84.2 |
| 2008 | 83.2 |

The coefficients of the least squares regression parabola $y = at^2 + bt + c$, where $y$ represents the retail sales (in billions of dollars) and $t$ represents the year, with $t = 4$ corresponding to 2004, can be found by solving the system

$$\begin{cases} 8674a + 1260b + 190c = 15{,}489.6 \\ 1260a + 190b + 30c = 2422.0. \\ 190a + 30b + 5c = 398.8 \end{cases}$$

(a) Use Cramer's Rule to solve the system and write the least squares regression parabola for the data.

(b) Use a graphing utility to graph the parabola with the data. How well does the model fit the data?

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**28. MODELING DATA**

The retail sales (in billions of dollars) of stores selling auto parts, accessories, and tires in the United States from 2004 through 2008 are shown in the table. (Source: U.S. Census Bureau)

| Year | Sales (in billions of dollars) |
|---|---|
| 2004 | 67.2 |
| 2005 | 71.2 |
| 2006 | 74.5 |
| 2007 | 76.7 |
| 2008 | 78.6 |

The coefficients of the least squares regression parabola $y = at^2 + bt + c$, where $y$ represents the retail sales (in billions of dollars) and $t$ represents the year, with $t = 4$ corresponding to 2004, can be found by solving the system

$$\begin{cases} 8674a + 1260b + 190c = 14{,}325.9 \\ 1260a + 190b + 30c = 2237.5. \\ 190a + 30b + 5c = 368.2 \end{cases}$$

(a) Use Cramer's Rule to solve the system and write the least squares regression parabola for the data.

(b) Use a graphing utility to graph the parabola with the data. How well does the model fit the data?

(c) Is this a good model for predicting retail sales in future years? Explain.

**Encoding a Message** **In Exercises 29 and 30, (a) write the uncoded 1 × 3 row matrices for the message, and then (b) encode the message using the encoding matrix.**

| *Message* | *Encoding Matrix* |
|---|---|
| ✓ **29.** TEXT ME AT WORK | $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$ |
| **30.** PLEASE SEND MONEY | $\begin{bmatrix} 4 & 2 & 1 \\ -3 & -3 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ |

**Encoding a Message** **In Exercises 31 and 32, write a cryptogram for the message using the matrix $A$.**

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$$

**31.** KEY UNDER RUG **32.** HAPPY BIRTHDAY

**Decoding a Message** **In Exercises 33–35, use $A^{-1}$ to decode the cryptogram.**

**33.** $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ 11 21 64 112 25 50 29
53 23 46 40 75 55 92

**34.** $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ 85 120 6 8 10 15 84 117 42
56 90 125 60 80 30 45 19 26

✓ **35.** $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & -2 \end{bmatrix}$ 3 18 21 31 29 13 −2 −1
4 −6 28 54 −3 4 12 16
8 1 6 6 0 27 −12 −39
15 −19 −27 5 10 5

**36.** ***Why you should learn it*** *(p. 548)* The following cryptogram was encoded with a 2 × 2 matrix.

8 21 −15 −10 −13 −13 5 10 5 25
5 19 −1 6 20 40 −18 −18 1 16

The last word of the message is _RON. What is the message?

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## Conclusions

**True or False?** **In Exercises 37 and 38, determine whether the statement is true or false. Justify your answer.**

**37.** Cramer's Rule cannot be used to solve a system of linear equations when the determinant of the coefficient matrix is zero.

**38.** In a system of linear equations, when the determinant of the coefficient matrix is zero, the system has no solution.

**39.** **Think About It** Describe a way to use an invertible $n \times n$ matrix to encode a message that is converted to numbers and partitioned into uncoded column matrices.

**40. CAPSTONE** Consider the system of linear equations

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

where $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, and $c_2$ represent real numbers. What must be true about the lines represented by the equations when

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0?$$

## Cumulative Mixed Review

**Equation of a Line** **In Exercises 41–44, find the general form of the equation of the line that passes through the two points.**

**41.** $(-1, 5), (7, 3)$
**42.** $(0, -6), (-2, 10)$
**43.** $(3, -3), (10, -1)$
**44.** $(-4, 12), (4, 2)$

# 7 Chapter Summary

| | What did you learn? | Explanation and Examples | Review Exercises |
|---|---|---|---|
| 7.1 | **Use the methods of substitution and graphing to solve systems of equations in two variables *(p. 470),* and use systems of equations to model and solve real-life problems *(p. 475).*** | **Substitution:** (1) Solve one of the equations for one variable. (2) Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable and (3) solve the equation. (4) Back-substitute the value(s) obtained in Step 3 into the expression obtained in Step 1 to find the value(s) of the other variable. (5) Check the solution(s).<br>**Graphing:** (1) Solve both equations for $y$ in terms of $x$. (2) Use a graphing utility to graph both equations. (3) Use the *intersect* feature or the *zoom* and *trace* features of the graphing utility to approximate the point(s) of intersection of the graphs. (4) Check the solution(s). | 1–22 |
| 7.2 | **Use the method of elimination to solve systems of linear equations in two variables *(p. 480),* and graphically interpret the number of solutions of a system of linear equations in two variables *(p. 482).*** | **Elimination:** (1) Obtain coefficients for $x$ (or $y$) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants. (2) Add the equations to eliminate one variable and solve the resulting equation. (3) Back-substitute the value obtained in Step 2 into either of the original equations and solve for the other variable. (4) Check the solution(s). | 23–38 |
| | **Use systems of linear equations in two variables to model and solve real-life problems *(p. 484).*** | A system of linear equations in two variables can be used to find the airspeed of an airplane and the speed of the wind. (See Example 6.) | 39–42 |
| 7.3 | **Use back-substitution to solve linear systems in row-echelon form *(p. 489).*** | $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases} \rightarrow \begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$ | 43, 44 |
| | **Use Gaussian elimination to solve systems of linear equations *(p. 490).*** | Use elementary row operations to convert a system of linear equations to row-echelon form. (1) Interchange two equations. (2) Multiply one of the equations by a nonzero constant. (3) Add a multiple of one equation to another equation. | 45–50 |
| | **Solve nonsquare systems of linear equations *(p. 493).*** | In a nonsquare system, the number of equations differs from the number of variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables. | 51, 52 |
| | **Graphically interpret three-variable linear systems *(p. 494).*** | The graph of a system of three linear equations in three variables consists of three planes. When they intersect in a single point, the system has exactly one solution. When they have no point in common, the system has no solution. When they intersect in a line or a plane, the system has infinitely many solutions (see Figures 7.16–7.20). | 53, 54 |
| | **Use systems of linear equations to write partial fraction decompositions of rational expressions *(p. 495).*** | $\frac{9}{x^3 - 6x^2} = \frac{9}{x^2(x - 6)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 6}$ | 55–60 |
| | **Use systems of linear equations in three or more variables to model and solve real-life problems *(p. 498).*** | A system of linear equations in three variables can be used to find the position equation of an object that is moving in a (vertical) line with constant acceleration. (See Example 8.) | 61–64 |

| | What did you learn? | Explanation and Examples | Review Exercises |
|---|---|---|---|
| 7.4 | **Write matrices and identify their dimensions *(p. 504)*, and perform elementary row operations on matrices *(p. 506)*.** | **Elementary Row Operations**<br>**1.** Interchange two rows.<br>**2.** Multiply a row by a nonzero constant.<br>**3.** Add a multiple of a row to another row. | 65–80 |
| | **Use matrices and Gaussian elimination to solve systems of linear equations *(p. 507)*.** | Write the augmented matrix of the system. Use elementary row operations to rewrite the augmented matrix in row-echelon form. Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution. | 81–88 |
| | **Use matrices and Gauss-Jordan elimination to solve systems of linear equations *(p. 511)*.** | Gauss-Jordan elimination continues the reduction process on a matrix in row-echelon form until a reduced row-echelon form is obtained. (See Example 8.) | 89–98 |
| 7.5 | **Decide whether two matrices are equal *(p. 518)*.** | Two matrices are equal when they have the same dimension and all of their corresponding entries are equal. | 99–102 |
| | **Add and subtract matrices and multiply matrices by scalars *(p. 519)*, multiply two matrices *(p. 522)*, and use matrix operations to model and solve real-life problems *(p. 525)*.** | **1.** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of dimension $m \times n$ and let $c$ be a scalar.<br>$A + B = [a_{ij} + b_{ij}] \qquad cA = [ca_{ij}]$<br>**2.** Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. The product $AB$ is an $m \times p$ matrix given by $AB = [c_{ij}]$, where<br>$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}$. | 103–126 |
| 7.6 | **Verify that two matrices are inverses of each other *(p. 532)*, and use Gauss-Jordan elimination to find inverses of matrices *(p. 533)*.** | Write the $n \times 2n$ matrix $[A \ \vdots \ I]$. Row reduce $A$ to $I$ using elementary row operations. The result will be the matrix $[I \ \vdots \ A^{-1}]$. | 127–136 |
| | **Use a formula to find inverses of 2 × 2 matrices *(p. 536)*.** | If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then<br>$A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. | 137–140 |
| | **Use inverse matrices to solve systems of linear equations *(p. 537)*.** | If $A$ is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$. | 141–150 |
| 7.7 | **Find the determinants of 2 × 2 matrices *(p. 541)*.** | $\det(A) = \lvert A \rvert = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ | 151–154 |
| | **Find minors and cofactors of square matrices *(p. 543)*, and find the determinants of square matrices *(p. 544)*.** | The determinant of a square matrix $A$ (of dimension $2 \times 2$ or greater) is the sum of the entries in any row or column of $A$ multiplied by their respective cofactors. | 155–166 |
| 7.8 | **Use determinants to find areas of triangles *(p. 548)* and to decide whether points are collinear *(p. 549)*, use Cramer's Rule to solve systems of linear equations *(p. 550)*, and use matrices to encode and decode messages *(p. 553)*.** | If a system of $n$ linear equations in $n$ variables has a coefficient matrix $A$ with a nonzero determinant $\lvert A \rvert$, then the solution of the system is<br>$x_1 = \dfrac{\lvert A_1 \rvert}{\lvert A \rvert}, x_2 = \dfrac{\lvert A_2 \rvert}{\lvert A \rvert}, \ . \ . \ . \ , x_n = \dfrac{\lvert A_n \rvert}{\lvert A \rvert}$<br>where the $i$th column of $A_i$ is the column of constants in the system of equations. | 167–193 |

# 7 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

## 7.1

**Solving a System by Substitution** **In Exercises 1–10, solve the system by the method of substitution.**

**1.** $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$

**2.** $\begin{cases} 2x - 3y = 3 \\ x - y = 0 \end{cases}$

**3.** $\begin{cases} 4x - y = 1 \\ 8x + y = 17 \end{cases}$

**4.** $\begin{cases} 10x + 6y = -14 \\ x + 9y = -7 \end{cases}$

**5.** $\begin{cases} 0.5x + y = 0.75 \\ 1.25x - 4.5y = -2.5 \end{cases}$

**6.** $\begin{cases} -x + \frac{2}{5}y = \frac{3}{5} \\ -x + \frac{1}{5}y = -\frac{4}{5} \end{cases}$

**7.** $\begin{cases} x^2 - y^2 = 9 \\ x - y = 1 \end{cases}$

**8.** $\begin{cases} x^2 + y^2 = 169 \\ 3x + 2y = 39 \end{cases}$

**9.** $\begin{cases} y = 2x^2 \\ y = x^4 - 2x^2 \end{cases}$

**10.** $\begin{cases} x = y + 3 \\ x = y^2 + 1 \end{cases}$

**Solving a System of Equations Graphically** **In Exercises 11–18, use a graphing utility to approximate all points of intersection of the graphs of the equations in the system. Verify your solutions by checking them in the original system.**

**11.** $\begin{cases} 5x + 6y = 7 \\ -x - 4y = 0 \end{cases}$

**12.** $\begin{cases} 8x - 3y = -3 \\ 2x + 5y = 28 \end{cases}$

**13.** $\begin{cases} y^2 - 4x = 0 \\ x + y = 0 \end{cases}$

**14.** $\begin{cases} y^2 - x = -1 \\ y + 2x = 5 \end{cases}$

**15.** $\begin{cases} y = 3 - x^2 \\ y = 2x^2 + x + 1 \end{cases}$

**16.** $\begin{cases} y = 2x^2 - 4x + 1 \\ y = x^2 - 4x + 3 \end{cases}$

**17.** $\begin{cases} y = 2(6 - x) \\ y = 2^{x-2} \end{cases}$

**18.** $\begin{cases} y = \ln(x + 2) + 1 \\ x + y = 0 \end{cases}$

**19.** **Finance** You invest \$5000 in a greenhouse. The planter, potting soil, and seed for each plant cost \$6.43, and the selling price of each plant is \$12.68. How many plants must you sell to break even?

**20.** **Finance** You are offered two sales jobs. One company offers an annual salary of \$55,000 plus a year-end bonus of 1.5% of your total sales. The other company offers an annual salary of \$52,000 plus a year-end bonus of 2% of your total sales. How much would you have to sell in a year to make the second offer the better offer?

**21.** **Geometry** The perimeter of a rectangle is 480 meters and its length is 1.5 times its width. Find the dimensions of the rectangle.

**22.** **Geometry** The perimeter of a rectangle is 68 feet and its width is $\frac{8}{9}$ times its length. Find the dimensions of the rectangle.

## 7.2

**Solving a System by Elimination** **In Exercises 23–32, solve the system by the method of elimination.**

**23.** $\begin{cases} 2x - y = 2 \\ 6x + 8y = 39 \end{cases}$

**24.** $\begin{cases} 40x + 30y = 24 \\ 20x - 50y = -14 \end{cases}$

**25.** $\begin{cases} 0.2x + 0.3y = 0.14 \\ 0.4x + 0.5y = 0.20 \end{cases}$

**26.** $\begin{cases} 12x + 42y = -17 \\ 30x - 18y = 19 \end{cases}$

**27.** $\begin{cases} \frac{1}{5}x + \frac{3}{10}y = \frac{7}{50} \\ \frac{2}{5}x + \frac{1}{2}y = \frac{1}{5} \end{cases}$

**28.** $\begin{cases} \frac{5}{12}x - \frac{3}{4}y = \frac{25}{4} \\ -x + \frac{7}{8}y = -\frac{38}{5} \end{cases}$

**29.** $\begin{cases} 3x - 2y = 0 \\ 3x + 2(y + 5) = 10 \end{cases}$

**30.** $\begin{cases} 7x + 12y = 63 \\ 2x + 3y = 15 \end{cases}$

**31.** $\begin{cases} 1.25x - 2y = 3.5 \\ 5x - 8y = 14 \end{cases}$

**32.** $\begin{cases} 1.5x + 2.5y = 8.5 \\ 6x + 10y = 24 \end{cases}$

**Solving a System Graphically** **In Exercises 33–38, use a graphing utility to graph the lines in the system. Use the graphs to determine whether the system is consistent or inconsistent. If the system is consistent, determine the solution. Verify your results algebraically.**

**33.** $\begin{cases} 3x + 2y = 0 \\ x - y = 4 \end{cases}$

**34.** $\begin{cases} x + y = 6 \\ -2x - 2y = -12 \end{cases}$

**35.** $\begin{cases} \frac{1}{4}x - \frac{1}{5}y = 2 \\ -5x + 4y = 8 \end{cases}$

**36.** $\begin{cases} \frac{7}{2}x - 7y = -1 \\ -x + 2y = 4 \end{cases}$

**37.** $\begin{cases} 2x - 2y = 8 \\ 4x - 1.5y = -5.5 \end{cases}$

**38.** $\begin{cases} -x + 3.2y = 10.4 \\ -2x - 9.6y = 6.4 \end{cases}$

**Supply and Demand** **In Exercises 39 and 40, find the point of equilibrium of the demand and supply equations.**

| | *Demand* | *Supply* |
|---|---|---|
| **39.** | $p = 37 - 0.0002x$ | $p = 22 + 0.00001x$ |
| **40.** | $p = 120 - 0.0001x$ | $p = 45 + 0.0002x$ |

**41.** **Aerodynamics** Two planes leave Pittsburgh and Philadelphia at the same time, each going to the other city. One plane flies 25 miles per hour faster than the other. Find the airspeed of each plane given that the cities are 275 miles apart and the planes pass each other after 40 minutes of flying time.

**42.** **Economics** A total of \$46,000 is invested in two corporate bonds that pay 6.75% and 7.25% simple interest. The investor wants an annual interest income of \$3245 from the investments. What is the most that can be invested in the 6.75% bond?

## 7.3

**Using Back-Substitution** **In Exercises 43 and 44, use back-substitution to solve the system of linear equations.**

**43.** $\begin{cases} x - 4y + 3z = 3 \\ -y + z = -1 \\ z = -5 \end{cases}$ **44.** $\begin{cases} x - 7y + 8z = -14 \\ y - 9z = 26 \\ z = -3 \end{cases}$

**Solving a System of Linear Equations** **In Exercises 45–52, solve the system of linear equations and check any solution algebraically.**

**45.** $\begin{cases} x + 3y - z = 13 \\ 2x - 5z = 23 \\ 4x - y - 2z = 14 \end{cases}$ **46.** $\begin{cases} x + y + z = 2 \\ -x + 3y + 2z = 8 \\ 4x + y = 4 \end{cases}$

**47.** $\begin{cases} x - 2y + z = -6 \\ 2x - 3y = -7 \\ -x + 3y - 3z = 11 \end{cases}$ **48.** $\begin{cases} 2x + 6z = -9 \\ 3x - 2y + 11z = -16 \\ 3x - y + 7z = -11 \end{cases}$

**49.** $\begin{cases} x - 2y + 3z = -5 \\ 2x + 4y + 5z = 1 \\ x + 2y + z = 0 \end{cases}$ **50.** $\begin{cases} x - 2y + z = 5 \\ 2x + 3y + z = 5 \\ x + y + 2z = 3 \end{cases}$

**51.** $\begin{cases} 5x - 12y + 7z = 16 \\ 3x - 7y + 4z = 9 \end{cases}$ **52.** $\begin{cases} 2x + 5y - 19z = 34 \\ 3x + 8y - 31z = 54 \end{cases}$

**Sketching a Plane** **In Exercises 53 and 54, sketch the plane represented by the linear equation. Then list four points that lie in the plane.**

**53.** $2x - 4y + z = 8$ **54.** $3x + 3y - z = 9$

**Partial Fraction Decomposition** **In Exercises 55–60, write the partial fraction decomposition for the rational expression. Check your result algebraically by combining the fractions, and check your result graphically by using a graphing utility to graph the rational expression and the partial fractions in the same viewing window.**

**55.** $\dfrac{4 - x}{x^2 + 6x + 8}$ **56.** $\dfrac{-x}{x^2 + 3x + 2}$

**57.** $\dfrac{x^2 + 2x}{x^3 - x^2 + x - 1}$ **58.** $\dfrac{3x^3 + 4x}{x^4 + 2x^2 + 1}$

**59.** $\dfrac{x^2 + 3x - 3}{x^3 + 2x^2 + x + 2}$ **60.** $\dfrac{2x^2 - x + 7}{x^4 + 8x^2 + 16}$

**Data Analysis: Curve-Fitting** **In Exercises 61 and 62, find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points. To verify your result, use a graphing utility to plot the points and graph the parabola.**

**61.** $(-1, -4), (1, -2), (2, 5)$ **62.** $(-1, 0), (1, 4), (2, 3)$

**63. Physical Education** Pebble Beach Golf Links in Pebble Beach, California is an 18-hole course that consists of par-3 holes, par-4 holes, and par-5 holes. There are two more par-4 holes than twice the number of par-5 holes, and the number of par-3 holes is equal to the number of par-5 holes. Find the number of par-3, par-4, and par-5 holes on the course. (Source: Pebble Beach Resorts)

**64. Economics** An inheritance of \$40,000 is divided among three investments yielding \$3500 in interest per year. The interest rates for the three investments are 7%, 9%, and 11%. Find the amount of each investment if the second and third are \$3000 and \$5000 less than the first, respectively.

## 7.4

**Dimension of a Matrix** **In Exercises 65–68, determine the dimension of the matrix.**

**65.** $\begin{bmatrix} -3 \\ 1 \\ 10 \end{bmatrix}$ **66.** $\begin{bmatrix} 3 & -1 & 0 & 6 \\ -2 & 7 & 1 & 4 \end{bmatrix}$

**67.** $[3]$ **68.** $[6 \quad 7 \quad -5 \quad 0 \quad -8]$

**Writing an Augmented Matrix** **In Exercises 69–72, write the augmented matrix for the system of linear equations.**

**69.** $\begin{cases} 6x - 7y = 11 \\ -2x + 5y = -1 \end{cases}$ **70.** $\begin{cases} -x + y = 12 \\ 10x - 4y = -90 \end{cases}$

**71.** $\begin{cases} 8x - 7y + 4z = 12 \\ 3x - 5y + 2z = 20 \\ 5x + 3y - 3z = 26 \end{cases}$ **72.** $\begin{cases} 3x - 5y + z = 25 \\ -4x - 2z = -14 \\ 6x + y = 15 \end{cases}$

**Writing a System of Equations** **In Exercises 73 and 74, write the system of linear equations represented by the augmented matrix. (Use the variables *x*, *y*, *z*, and *w*, if applicable.)**

**73.** $\left[\begin{array}{rrrcr} 5 & 1 & 7 & \vdots & -9 \\ 4 & 2 & 0 & \vdots & 10 \\ 9 & 4 & 2 & \vdots & 3 \end{array}\right]$

**74.** $\left[\begin{array}{rrrrcr} 13 & 16 & 7 & 3 & \vdots & 2 \\ 1 & 21 & 8 & 5 & \vdots & 12 \\ 4 & 10 & -4 & 3 & \vdots & -1 \end{array}\right]$

**Using Gaussian Elimination** **In Exercises 75 and 76, write the matrix in row-echelon form. Remember that the row-echelon form of a matrix is not unique.**

**75.** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$ **76.** $\begin{bmatrix} 4 & 8 & 16 \\ 3 & -1 & 2 \\ -2 & 10 & 12 \end{bmatrix}$

**Using a Graphing Utility** **In Exercises 77–80, use the matrix capabilities of a graphing utility to write the matrix in reduced row-echelon form.**

**77.** $\begin{bmatrix} 3 & -2 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{bmatrix}$

**78.** $\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 2 & 8 & 0 & 0 & 1 \end{bmatrix}$

**79.** $\begin{bmatrix} 1.5 & 3.6 & 4.2 \\ 0.2 & 1.4 & 1.8 \\ 2.0 & 4.4 & 6.4 \end{bmatrix}$

**80.** $\begin{bmatrix} 4.1 & 8.3 & 1.6 \\ 3.2 & -1.7 & 2.4 \\ -2.3 & 1.0 & 1.2 \end{bmatrix}$

**Gaussian Elimination with Back-Substitution** **In Exercises 81–88, use matrices to solve the system of equations, if possible. Use Gaussian elimination with back-substitution.**

**81.** $\begin{cases} 5x + 4y = 2 \\ -x + y = -22 \end{cases}$

**82.** $\begin{cases} 2x - 5y = 2 \\ 3x - 7y = 1 \end{cases}$

**83.** $\begin{cases} 0.3x - 0.1y = -0.13 \\ 0.2x - 0.3y = -0.25 \end{cases}$

**84.** $\begin{cases} 0.2x - 0.1y = 0.07 \\ 0.4x - 0.5y = -0.01 \end{cases}$

**85.** $\begin{cases} 2x + 3y + 3z = 3 \\ 6x + 6y + 12z = 13 \\ 12x + 9y - z = 2 \end{cases}$

**86.** $\begin{cases} x + 2y + 6z = 1 \\ 2x + 5y + 15z = 4 \\ 3x + y + 3z = -6 \end{cases}$

**87.** $\begin{cases} x + 2y - z = 1 \\ y + z = 0 \end{cases}$

**88.** $\begin{cases} x - y + 4z - w = 4 \\ x + 3y - 2z + w = -4 \\ y - z + w = -3 \\ 2x + z + w = 0 \end{cases}$

**Gauss-Jordan Elimination** **In Exercises 89–94, use matrices to solve the system of equations, if possible. Use Gauss-Jordan elimination.**

**89.** $\begin{cases} -x + y + 2z = 1 \\ 2x + 3y + z = -2 \\ 5x + 4y + 2z = 4 \end{cases}$

**90.** $\begin{cases} 4x + 4y + 4z = 5 \\ 4x - 2y - 8z = 1 \\ 5x + 3y + 8z = 6 \end{cases}$

**91.** $\begin{cases} x + y + 2z = 4 \\ x - y + 4z = 1 \\ 2x - y + 2z = 1 \end{cases}$

**92.** $\begin{cases} x + y + 4z = 0 \\ 2x + y + 2z = 0 \\ -x + y - 2z = -1 \end{cases}$

**93.** $\begin{cases} x + 2y - z = 3 \\ x - y - z = -3 \\ 2x + y + 3z = 10 \end{cases}$

**94.** $\begin{cases} x - 3y + z = 2 \\ 3x - y - z = -6 \\ -x + y - 3z = -2 \end{cases}$

**Using a Graphing Utility** **In Exercises 95–98, use the matrix capabilities of a graphing utility to reduce the augmented matrix corresponding to the system of equations, and solve the system.**

**95.** $\begin{cases} x + 2y - z = 7 \\ -y - z = 4 \\ 4x - z = 16 \end{cases}$

**96.** $\begin{cases} 3x + 6z = 0 \\ -2x + y = 5 \\ y + 2z = 3 \end{cases}$

**97.** $\begin{cases} 3x - y + 5z - 2w = -44 \\ x + 6y + 4z - w = 1 \\ 5x - y + z + 3w = -15 \\ 4y - z - 8w = 58 \end{cases}$

**98.** $\begin{cases} 4x + 12y + 2z = 20 \\ x + 6y + 4z = 12 \\ x + 6y + z = 8 \\ -2x - 10y - 2z = -10 \end{cases}$

**7.5**

**Equality of Matrices** **In Exercises 99–102, find $x$ and $y$.**

**99.** $\begin{bmatrix} -1 & x \\ y & 9 \end{bmatrix} = \begin{bmatrix} -1 & 12 \\ -7 & 9 \end{bmatrix}$

**100.** $\begin{bmatrix} -1 & 0 \\ x & 5 \\ -4 & y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 8 & 5 \\ -4 & 0 \end{bmatrix}$

**101.** $\begin{bmatrix} x+3 & 4 & -4y \\ 0 & -3 & 2 \\ -2 & y+5 & 6x \end{bmatrix} = \begin{bmatrix} 5x-1 & 4 & -44 \\ 0 & -3 & 2 \\ -2 & 16 & 6 \end{bmatrix}$

**102.** $\begin{bmatrix} -9 & 4 & 2 & -5 \\ 0 & -3 & 7 & -4 \\ 6 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 4 & x-10 & -5 \\ 0 & -3 & 7 & 2y \\ \frac{1}{2}x & -1 & 1 & 0 \end{bmatrix}$

**Operations with Matrices** **In Exercises 103–106, find, if possible, (a) $A + B$, (b) $A - B$, (c) $2A$, and (d) $A + 3B$.**

**103.** $A = \begin{bmatrix} 7 & 3 \\ -1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & -20 \\ 14 & -3 \end{bmatrix}$

**104.** $A = \begin{bmatrix} -11 & 16 & 19 \\ -7 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 0 \\ 8 & -4 \\ -2 & 10 \end{bmatrix}$

**105.** $A = \begin{bmatrix} 6 & 0 & 7 \\ 5 & -1 & 2 \\ 3 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 5 & 1 \\ -4 & 8 & 6 \\ 2 & -1 & 1 \end{bmatrix}$

**106.** $A = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 5 & 5 \\ 1 & 1 & 1 \end{bmatrix}$

**Using the Distributive Property** **In Exercises 107–110, evaluate the expression. If it is not possible, explain why.**

**107.** $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & -4 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 & -6 \\ 0 & -2 & 5 \end{bmatrix}$

**108.** $-4\begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8\begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$

**109.** $-1\begin{bmatrix} 8 & -1 \\ -2 & 4 \end{bmatrix} - 5\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 7 & -8 \\ 4 & 3 \end{bmatrix}$

**110.** $6\left(\begin{bmatrix} -4 & -1 & -3 & 4 \\ 2 & -5 & 7 & -10 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 13 & -7 \\ 14 & -3 & 8 & -1 \end{bmatrix}\right)$

**Operations with Matrices** **In Exercises 111 and 112, use the matrix capabilities of a graphing utility to evaluate the expression.**

**111.** $3\begin{bmatrix} 8 & -2 & 5 \\ 1 & 3 & -1 \end{bmatrix} + 6\begin{bmatrix} 4 & -2 & -3 \\ 2 & 7 & 6 \end{bmatrix}$

**112.** $-5\begin{bmatrix} 2.7 & 0.2 \\ 7.3 & -2.9 \\ 8.6 & 2.1 \end{bmatrix} + 4\begin{bmatrix} 4.4 & -2.3 \\ 6.6 & 11.6 \\ -1.5 & 3.9 \end{bmatrix}$

**Solving a Matrix Equation** **In Exercises 113–116, solve for $X$ when**

$$A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}.$$

**113.** $X = 3A - 2B$

**114.** $6X = 4A + 3B$

**115.** $3X + 2A = B$

**116.** $2A - 5B = 3X$

**Finding the Product of Two Matrices** **In Exercises 117–120, find $AB$, if possible.**

**117.** $A = \begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$

**118.** $A = \begin{bmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 5 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

**119.** $A = \begin{bmatrix} 6 & -5 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix}$

**120.** $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 1 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \\ 0 & 6 & 2 \end{bmatrix}$

**Operations with Matrices** **In Exercises 121–124, use the matrix capabilities of a graphing utility to evaluate the expression.**

**121.** $\begin{bmatrix} 4 & 1 \\ 11 & -7 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 & 6 \\ 2 & -2 & -2 \end{bmatrix}$

**122.** $\begin{bmatrix} -2 & 3 & 10 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 2 \\ 3 & 2 \end{bmatrix}$

**123.** $\begin{bmatrix} 2 & 1 \\ 6 & 0 \end{bmatrix} \left( \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & 4 \end{bmatrix} \right)$

**124.** $\begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & -3 \end{bmatrix}$

**125.** **Business** At a dairy mart, the numbers of gallons of fat free, 2%, and whole milk sold on Friday, Saturday, and Sunday of a particular week are given by the following matrix.

$$A = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{ccc|l} \text{Fat free milk} & \text{2\% milk} & \text{Whole milk} & \\ 40 & 64 & 52 & \text{Friday} \\ 60 & 82 & 76 & \text{Saturday} \\ 76 & 96 & 84 & \text{Sunday} \end{array}$$

A second matrix gives the selling price per gallon and the profit per gallon for each of the three types of milk sold by the dairy mart.

$$B = \begin{array}{ccl} \text{Selling price per gallon} & \text{Profit per gallon} & \\ 3.25 & 0.25 & \text{Fat free milk} \\ 3.35 & 0.30 & \text{2\% milk} \\ 3.49 & 0.35 & \text{Whole milk} \end{array}$$

(a) Find $AB$. What is the meaning of $AB$ in the context of the situation?

(b) Find the dairy mart's profit for Friday through Sunday.

**126.** **Business** The pay-as-you-go charges (in dollars per minute) of two cellular telephone companies for calls inside the coverage area, regional roaming calls, and calls outside the coverage area are represented by $C$.

$$C = \begin{array}{ccl} \multicolumn{2}{c}{\overbrace{\hspace{6em}}^{\text{Company}}} & \\ A & B & \\ 0.07 & 0.095 & \text{Inside} \\ 0.10 & 0.08 & \text{Regional roaming} \\ 0.28 & 0.25 & \text{Outside} \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{Coverage area}$$

Each month, you plan to use 120 minutes on calls inside the coverage area, 80 minutes on regional roaming calls, and 20 minutes on calls outside the coverage area.

(a) Write a matrix $T$ that represents the times spent on the phone for each type of call.

(b) Compute $TC$ and interpret the result.

7.6

**The Inverse of a Matrix** **In Exercises 127 and 128, show that $B$ is the inverse of $A$.**

**127.** $A = \begin{bmatrix} -4 & -1 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 \\ 7 & 4 \end{bmatrix}$

**128.** $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 6 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -3 & 1 \\ 3 & 3 & -1 \\ 2 & 4 & -1 \end{bmatrix}$

**Finding the Inverse of a Matrix** **In Exercises 129–132, find the inverse of the matrix (if it exists).**

**129.** $\begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix}$

**130.** $\begin{bmatrix} -3 & -5 \\ 2 & 3 \end{bmatrix}$

**131.** $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$

**132.** $\begin{bmatrix} 0 & -2 & 1 \\ -5 & -2 & -3 \\ 7 & 3 & 4 \end{bmatrix}$

**Finding the Inverse of a Matrix** **In Exercises 133–136, use the matrix capabilities of a graphing utility to find the inverse of the matrix (if it exists).**

**133.** $\begin{bmatrix} 2 & 6 \\ 3 & -6 \end{bmatrix}$

**134.** $\begin{bmatrix} 3 & -10 \\ 4 & 2 \end{bmatrix}$

**135.** $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ **136.** $\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -2 \\ 1 & 2 & -4 \end{bmatrix}$

**Finding the Inverse of a 2 × 2 Matrix** **In Exercises 137–140, use the formula on page 536 to find the inverse of the 2 × 2 matrix.**

**137.** $\begin{bmatrix} -7 & 2 \\ -8 & 2 \end{bmatrix}$ **138.** $\begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix}$

**139.** $\begin{bmatrix} -1 & 10 \\ 2 & 20 \end{bmatrix}$ **140.** $\begin{bmatrix} -6 & -5 \\ 3 & 3 \end{bmatrix}$

**Solving a System of Equations Using an Inverse** **In Exercises 141–146, use an inverse matrix to solve (if possible) the system of linear equations.**

**141.** $\begin{cases} -x + 4y = 8 \\ 2x - 7y = -5 \end{cases}$ **142.** $\begin{cases} 2x + 3y = -10 \\ 4x - y = 1 \end{cases}$

**143.** $\begin{cases} 3x + 2y - z = 6 \\ x - y + 2z = -1 \\ 5x + y + z = 7 \end{cases}$ **144.** $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$

**145.** $\begin{cases} x + 2y + z - w = -2 \\ 2x + y + z + w = 1 \\ x - y - 3z = 0 \\ z + w = 1 \end{cases}$

**146.** $\begin{cases} x + y + z + w = 1 \\ x - y + 2z + w = -3 \\ y + w = 2 \\ x + w = 2 \end{cases}$

**Solving a System of Linear Equations** **In Exercises 147–150, use the matrix capabilities of a graphing utility to solve (if possible) the system of linear equations.**

**147.** $\begin{cases} x + 2y = -1 \\ 3x + 4y = -5 \end{cases}$ **148.** $\begin{cases} x + 3y = 23 \\ -6x + 2y = -18 \end{cases}$

**149.** $\begin{cases} -3x - 3y - 4z = 2 \\ y + z = -1 \\ 4x + 3y + 4z = -1 \end{cases}$

**150.** $\begin{cases} 2x + 3y - 4z = 1 \\ x - y + 2z = -4 \\ 3x + 7y - 10z = 0 \end{cases}$

## 7.7

**The Determinant of a Matrix** **In Exercises 151–154, find the determinant of the matrix.**

**151.** $\begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$ **152.** $\begin{bmatrix} -9 & 11 \\ 7 & -4 \end{bmatrix}$

**153.** $\begin{bmatrix} 50 & -30 \\ 10 & 5 \end{bmatrix}$ **154.** $\begin{bmatrix} 14 & -24 \\ 12 & -15 \end{bmatrix}$

**Finding the Minors and Cofactors of a Matrix** **In Exercises 155–158, find all (a) minors and (b) cofactors of the matrix.**

**155.** $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$ **156.** $\begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix}$

**157.** $\begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 0 \\ 1 & 8 & 6 \end{bmatrix}$ **158.** $\begin{bmatrix} 8 & 3 & 4 \\ 6 & 5 & -9 \\ -4 & 1 & 2 \end{bmatrix}$

**Finding a Determinant** **In Exercises 159–166, find the determinant of the matrix. Expand by cofactors on the row or column that appears to make the computations easiest.**

**159.** $\begin{bmatrix} -2 & 4 & 1 \\ -6 & 0 & 2 \\ 5 & 3 & 4 \end{bmatrix}$ **160.** $\begin{bmatrix} 4 & 7 & -1 \\ 2 & -3 & 4 \\ -5 & 1 & -1 \end{bmatrix}$

**161.** $\begin{bmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 3 \end{bmatrix}$ **162.** $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & 2 \\ -1 & -1 & 3 \end{bmatrix}$

**163.** $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ **164.** $\begin{bmatrix} 0 & 3 & 1 \\ 5 & -2 & 1 \\ 1 & 6 & 1 \end{bmatrix}$

**165.** $\begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 8 & 1 & 2 \\ 6 & 1 & 8 & 2 \\ 0 & 3 & -4 & 1 \end{bmatrix}$ **166.** $\begin{bmatrix} -5 & 6 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ -3 & 4 & -5 & 1 \\ 1 & 6 & 0 & 3 \end{bmatrix}$

## 7.8

**Finding the Area of a Figure** **In Exercises 167–174, use a determinant to find the area of the figure with the given vertices.**

**167.** $(1, 0), (5, 0), (5, 8)$

**168.** $(-4, 0), (4, 0), (0, 6)$

**169.** $\left(\frac{1}{2}, 1\right), \left(2, -\frac{5}{2}\right), \left(\frac{3}{2}, 1\right)$

**170.** $\left(\frac{3}{2}, 1\right), \left(4, -\frac{1}{2}\right), (4, 2)$

**171.** $(2, 4), (5, 6), (4, 1)$

**172.** $(-3, 2), (2, -3), (-4, -4)$

**173.** $(-2, -1), (4, 9), (-2, -9), (4, 1)$

**174.** $(-4, 8), (4, 0), (-4, 0), (4, -8)$

**Testing for Collinear Points** **In Exercises 175 and 176, use a determinant to determine whether the points are collinear.**

**175.** $(-1, 7), (3, -9), (-3, 15)$

**176.** $(0, -5), (2, 1), (4, 7)$

**Using Cramer's Rule** **In Exercises 177–184, use Cramer's Rule to solve (if possible) the system of equations.**

**177.** $\begin{cases} x + 2y = 5 \\ -x + y = 1 \end{cases}$ **178.** $\begin{cases} 2x - y = -10 \\ 3x + 2y = -1 \end{cases}$

**179.** $\begin{cases} 5x - 2y = 6 \\ -11x + 3y = -23 \end{cases}$ **180.** $\begin{cases} 3x + 8y = -7 \\ 9x - 5y = 37 \end{cases}$

**181.** $\begin{cases} -2x + 3y - 5z = -11 \\ 4x - y + z = -3 \\ -x - 4y + 6z = 15 \end{cases}$

**182.** $\begin{cases} 5x - 2y + z = 15 \\ 3x - 3y - z = -7 \\ 2x - y - 7z = -3 \end{cases}$

**183.** $\begin{cases} x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$

**184.** $\begin{cases} 14x - 21y - 7z = 10 \\ -4x + 2y - 2z = 4 \\ 56x - 21y + 7z = 5 \end{cases}$

**Comparing Solution Methods** **In Exercises 185 and 186, solve the system of equations using (a) Gaussian elimination and (b) Cramer's Rule. Which method do you prefer, and why?**

**185.** $\begin{cases} x - 3y + 2z = 5 \\ 2x + y - 4z = -1 \\ 2x + 4y + 2z = 3 \end{cases}$ **186.** $\begin{cases} x + 2y - z = -3 \\ 2x - y + z = -1 \\ 4x - 2y - z = 5 \end{cases}$

**Encoding a Message** **In Exercises 187 and 188, (a) write the uncoded $1 \times 3$ row matrices for the message, and then (b) encode the message using the encoding matrix.**

| | *Message* | *Encoding Matrix* |
|---|---|---|
| **187.** | LOOK OUT BELOW | $\begin{bmatrix} 2 & -2 & 0 \\ 3 & 0 & -3 \\ -6 & 2 & 3 \end{bmatrix}$ |
| **188.** | JUST DO IT | $\begin{bmatrix} 2 & 1 & 0 \\ -6 & -6 & -2 \\ 3 & 2 & 1 \end{bmatrix}$ |

**Decoding a Message** **In Exercises 189–192, use $A^{-1}$ to decode the cryptogram.**

**189.** $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -2 & 0 \\ 1 & -2 & 2 \end{bmatrix}$ 32 −46 37 9 −48
15 3 −14 10 −1
−6 2 −8 −22 −3

**190.** $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ 30 −7 30 5 10 80 37
34 16 40 −7 38 −3 8
36 16 −1 58 23 46 0

**191.** $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -2 \end{bmatrix}$ 21 −11 14 29 −11 −18
32 −6 −26 31 −19 −12
10 6 26 13 −11 −2 37
28 −8 5 13 36

**192.** $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & -2 \\ 1 & -1 & -2 \end{bmatrix}$ 9 15 −54 13 32 −26
8 −6 −14 −4 26 −70
−1 56 −38 28 27 −46
−13 27 −30 26 23 −48
25 4 −26 −11 31 −58
13 39 −34

**193. MODELING DATA**

The populations (in millions) of Florida for selected years from 2002 through 2008 are shown in the table. (Source: U.S. Census Bureau)

| Year | Population (in millions) |
|---|---|
| 2002 | 16.7 |
| 2004 | 17.3 |
| 2006 | 18.0 |
| 2008 | 18.3 |

The coefficients of the least squares regression line $y = at + b$, where $y$ is the population (in millions) and $t$ is the year, with $t = 2$ corresponding to 2002, can be found by solving the system

$$\begin{cases} 4b + 20a = 70.3 \\ 20b + 120a = 357 \end{cases}.$$

(a) Use Cramer's Rule to solve the system and find the least squares regression line.

(b) Use a graphing utility to graph the line from part (a).

(c) Use the graph from part (b) to estimate when the population of Florida will exceed 20 million.

(d) Use your regression equation to find algebraically when the population will exceed 20 million.

## *Conclusions*

**True or False?** **In Exercises 194 and 195, determine whether the statement is true or false. Justify your answer.**

**194.** Solving a system of equations graphically will always give an exact solution.

**195.** $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + c_1 & a_{32} + c_2 & a_{33} + c_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**196.** What is the relationship between the three elementary row operations performed on an augmented matrix and the operations that lead to equivalent systems of equations?

**197.** Under what conditions does a matrix have an inverse?

## 7 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

**Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.**

**In Exercises 1–3, solve the system by the method of substitution. Check your solution graphically.**

**1.** $\begin{cases} x - y = 6 \\ 3x + 5y = 2 \end{cases}$ **2.** $\begin{cases} y = x - 1 \\ y = (x-1)^3 \end{cases}$ **3.** $\begin{cases} 4x - y^2 = 7 \\ x - y = 3 \end{cases}$

**In Exercises 4–6, solve the system by the method of elimination.**

**4.** $\begin{cases} 2x + 5y = -11 \\ 5x - y = 19 \end{cases}$ **5.** $\begin{cases} 3x - 2y + z = 0 \\ 6x + 2y + 3z = -2 \\ 3x - 4y + 5z = 5 \end{cases}$ **6.** $\begin{cases} x - 4y - z = 3 \\ 2x - 5y + z = 0 \\ 3x - 3y + 2z = -1 \end{cases}$

**7.** Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(0, 6)$, $(-2, 2)$, and $\left(3, \frac{9}{2}\right)$.

**In Exercises 8 and 9, write the partial fraction decomposition for the rational expression.**

**8.** $\dfrac{5x - 2}{(x-1)^2}$ **9.** $\dfrac{x^3 + x^2 + x + 2}{x^4 + x^2}$

**In Exercises 10 and 11, use matrices to solve the system of equations, if possible.**

**10.** $\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$ **11.** $\begin{cases} 2x + 3y + z = 10 \\ 2x - 3y - 3z = 22 \\ 4x - 2y + 3z = -2 \end{cases}$

**12.** If possible, find (a) $A - B$, (b) $3A$, (c) $3A - 2B$, and (d) $AB$.

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -4 & -4 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 4 & 0 \\ 3 & 2 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

**13.** Find $A^{-1}$ for $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ and use $A^{-1}$ to solve the system at the right.

$$\begin{cases} -2x + 2y + 3z = 7 \\ x - y = -5 \\ y + 4z = -1 \end{cases}$$

System for 13

**In Exercises 14 and 15, find the determinant of the matrix.**

**14.** $\begin{bmatrix} -25 & 18 \\ 6 & -7 \end{bmatrix}$ **15.** $\begin{bmatrix} 4 & 0 & 3 \\ 1 & -8 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

**16.** Use determinants to find the area of the parallelogram shown at the right.

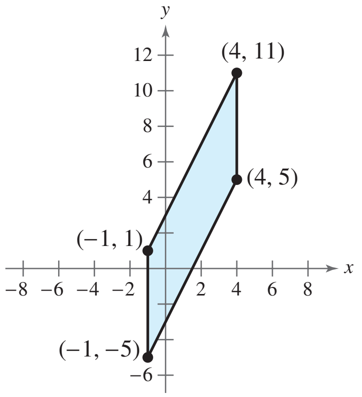


Figure for 16

**17.** Use Cramer's Rule to solve (if possible) $\begin{cases} 2x - 2y = 3 \\ x + 4y = -1 \end{cases}$.

**18.** The flow of traffic (in vehicles per hour) through a network of streets is shown at the right. Solve the system for the traffic flow represented by $x_i$, $i = 1, 2, 3, 4$, and 5.

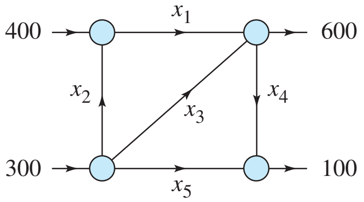


Figure for 18

# Proofs in Mathematics

An **indirect proof** can be useful in proving statements of the form "$p$ implies $q$." Recall that the conditional statement $p \rightarrow q$ is false only when $p$ is true and $q$ is false. To prove a conditional statement indirectly, assume that $p$ is true and $q$ is false. If this assumption leads to an impossibility, then you have proved that the conditional statement is true. An indirect proof is also called a **proof by contradiction.**

You can use an indirect proof to prove the conditional statement

"If $a$ is a positive integer and $a^2$ is divisible by 2, then $a$ is divisible by 2"

as follows. First, assume that $p$, "$a$ is a positive integer and $a^2$ is divisible by 2," is true and $q$, "$a$ is divisible by 2," is false. This means that $a$ is not divisible by 2. If so, then $a$ is odd and can be written as

$$a = 2n + 1$$

where $n$ is an integer.

$$a = 2n + 1 \qquad \text{Definition of an odd integer}$$

$$a^2 = 4n^2 + 4n + 1 \qquad \text{Square each side.}$$

$$a^2 = 2(2n^2 + 2n) + 1 \qquad \text{Distributive Property}$$

So, by the definition of an odd integer, $a^2$ is odd. This contradicts the assumption, and you can conclude that $a$ is divisible by 2.

## Example Using an Indirect Proof

Use an indirect proof to prove that $\sqrt{2}$ is an irrational number.

### Solution

Begin by assuming that $\sqrt{2}$ is *not* an irrational number. Then $\sqrt{2}$ can be written as the quotient of two integers $a$ and $b$ $(b \neq 0)$ that have no common factors.

$$\sqrt{2} = \frac{a}{b} \qquad \text{Assume that } \sqrt{2} \text{ is a rational number.}$$

$$2 = \frac{a^2}{b^2} \qquad \text{Square each side.}$$

$$2b^2 = a^2 \qquad \text{Multiply each side by } b^2.$$

This implies that 2 is a factor of $a^2$. So, 2 is also a factor of $a$, and $a$ can be written as $2c$, where $c$ is an integer.

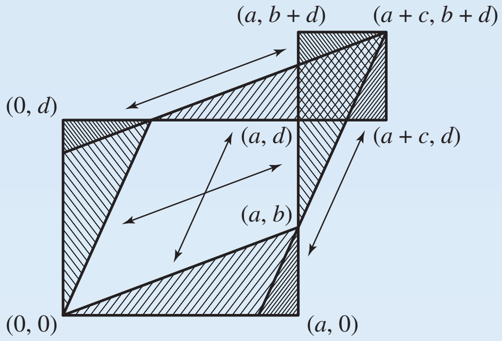
$$2b^2 = (2c)^2 \qquad \text{Substitute } 2c \text{ for } a.$$

$$2b^2 = 4c^2 \qquad \text{Simplify.}$$

$$b^2 = 2c^2 \qquad \text{Divide each side by 2.}$$

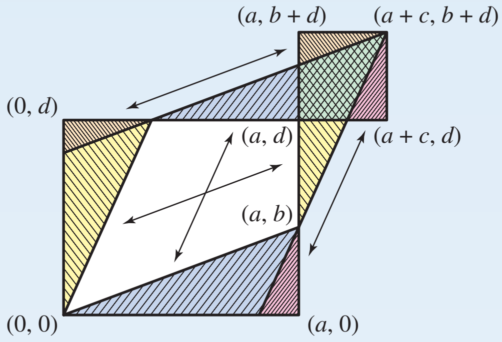
This implies that 2 is a factor of $b^2$ and also a factor of $b$. So, 2 is a factor of both $a$ and $b$. This contradicts the assumption that $a$ and $b$ have no common factors. So, you can conclude that $\sqrt{2}$ is an irrational number.

**Proofs without words** are pictures or diagrams that give a visual understanding of why a theorem or statement is true. They can also provide a starting point for writing a formal proof. The following proof shows that a 2 × 2 determinant is the area of a parallelogram.



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \|\text{▭}\| - \|\square\| = \|\text{▱}\|$$

The following is a color-coded version of the proof along with a brief explanation of why this proof works.



$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \|\text{▭}\| - \|\square\| = \|\text{▱}\|$$

Area of ▭ = Area of orange △ + Area of yellow △ + Area of blue △ +
Area of pink △ + Area of white quadrilateral

Area of □ = Area of orange △ + Area of pink △ + Area of green quadrilateral

Area of ▱ = Area of white quadrilateral + Area of blue △ + Area of yellow △ −
Area of green quadrilateral
= Area of ▭ − Area of □

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