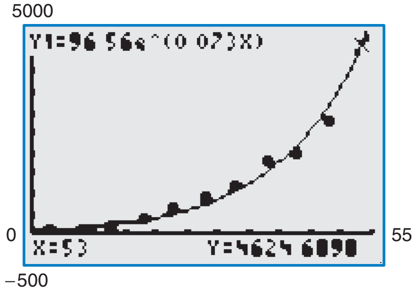
3 Exponential and Logarithmic Functions



Section 3.6, Example 4
Internal Revenue Service

3.1 Exponential Functions and Their Graphs

Exponential Functions

So far, this text has dealt mainly with **algebraic functions,** which include polynomial functions and rational functions. In this chapter you will study two types of nonalgebraic functions—*exponential functions* and *logarithmic functions*. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function f with base a** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

Note that in the definition of an exponential function, the base $a = 1$ is excluded because it yields

$$f(x) = 1^x = 1. \qquad \text{Constant function}$$

This is a constant function, not an exponential function.

You have already evaluated a^x for integer and rational values of x. For example, you know that

$$4^3 = 64 \quad \text{and} \quad 4^{1/2} = 2.$$

However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of

$$a^{\sqrt{2}} \left(\text{where } \sqrt{2} \approx 1.41421356\right)$$

as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \ldots .$$

Example 1 shows how to use a calculator to evaluate exponential functions.

Example 1 Evaluating Exponential Functions

Use a calculator to evaluate each function at the indicated value of x.

	Function	*Value*
a.	$f(x) = 2^x$	$x = -3.1$
b.	$f(x) = 2^{-x}$	$x = \pi$
c.	$f(x) = 0.6^x$	$x = \frac{3}{2}$
d.	$f(x) = 1.05^{2x}$	$x = 12$

Solution

	Function Value	*Graphing Calculator Keystrokes*	*Display*
a.	$f(-3.1) = 2^{-3.1}$	2 ^ (−) 3.1 ENTER	0.1166291
b.	$f(\pi) = 2^{-\pi}$	2 ^ (−) π ENTER	0.1133147
c.	$f\left(\frac{3}{2}\right) = (0.6)^{3/2}$	.6 ^ (3 ÷ 2) ENTER	0.4647580
d.	$f(12) = (1.05)^{2(12)}$	1.05 ^ (2 × 12) ENTER	3.2250999

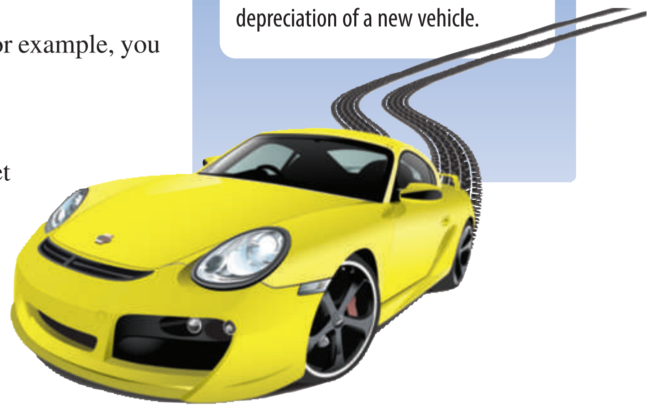
CHECKPOINT Now try Exercise 7.

What you should learn

- Recognize and evaluate exponential functions with base a.
- Graph exponential functions with base a.
- Recognize, evaluate, and graph exponential functions with base e.
- Use exponential functions to model and solve real-life problems.

Why you should learn it

Exponential functions are useful in modeling data that represent quantities that increase or decrease quickly. For instance, Exercise 74 on page 191 shows how an exponential function is used to model the depreciation of a new vehicle.



Technology Tip

When evaluating exponential functions with a calculator, remember to enclose fractional exponents in parentheses. Because the calculator follows the order of operations, parentheses are crucial in order to obtain the correct result.

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Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

Example 2 Graphs of $y = a^x$

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$

b. $g(x) = 4^x$

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.1. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
4^x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

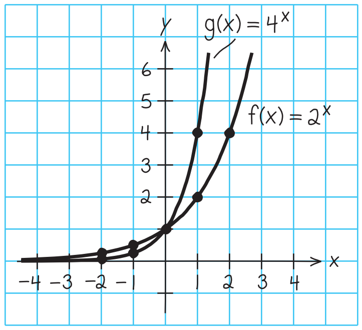


Figure 3.1

CHECKPOINT Now try Exercise 9.

Example 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graph of each function by hand.

a. $F(x) = 2^{-x}$

b. $G(x) = 4^{-x}$

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.2. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	-3	-2	-1	0	1	2
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
4^{-x}	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

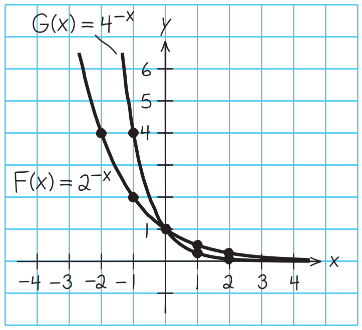


Figure 3.2

CHECKPOINT Now try Exercise 11.

The properties of exponents can also be applied to real-number exponents. For review, these properties are listed below.

1. $a^x a^y = a^{x+y}$ **2.** $\dfrac{a^x}{a^y} = a^{x-y}$ **3.** $a^{-x} = \dfrac{1}{a^x} = \left(\dfrac{1}{a}\right)^x$ **4.** $a^0 = 1$

5. $(ab)^x = a^x b^x$ **6.** $(a^x)^y = a^{xy}$ **7.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ **8.** $|a^2| = |a|^2 = a^2$

Study Tip

In Example 3, note that the functions $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents.

$$F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x \quad \text{and}$$

$$G(x) = 4^{-x} = \left(\frac{1}{4}\right)^x$$

Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \qquad \text{and} \qquad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y-axis) of the graph of f, as shown in Figure 3.3. The graphs of G and g have the same relationship, as shown in Figure 3.4.

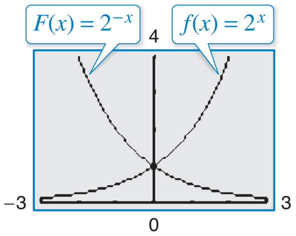


Figure 3.3

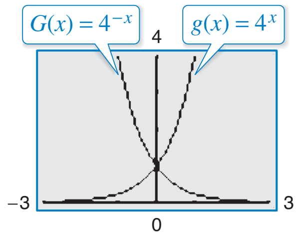


Figure 3.4

The graphs in Figures 3.3 and 3.4 are typical of the graphs of the exponential functions

$$f(x) = a^x \qquad \text{and} \qquad f(x) = a^{-x}.$$

They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous. The basic characteristics of these exponential functions are summarized below.

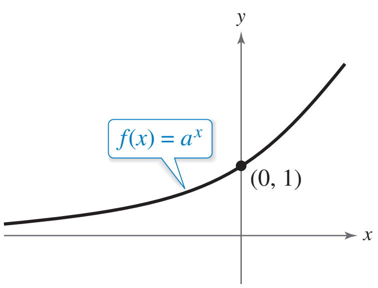
Library of Parent Functions: Exponential Function

The *parent exponential function*

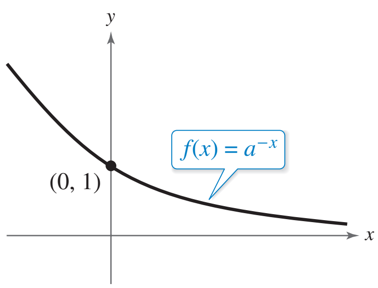
$$f(x) = a^x, \ a > 0, \ a \neq 1$$

is different from all the functions you have studied so far because the variable x is an *exponent*. A distinguishing characteristic of an exponential function is its rapid increase as x increases (for $a > 1$). Many real-life phenomena with patterns of rapid growth (or decline) can be modeled by exponential functions. The basic characteristics of the exponential function are summarized below and on the inside cover of this text.

Graph of $f(x) = a^x, \ a > 1$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercept: $(0, 1)$
Increasing on $(-\infty, \infty)$
x-axis is a horizontal asymptote
$(a^x \rightarrow 0$ as $x \rightarrow -\infty)$
Continuous



Graph of $f(x) = a^{-x}, \ a > 1$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Intercept: $(0, 1)$
Decreasing on $(-\infty, \infty)$
x-axis is a horizontal asymptote
$(a^{-x} \rightarrow 0$ as $x \rightarrow \infty)$
Continuous



Explore the Concept

Use a graphing utility to graph $y = a^x$ for $a = 3$, 5, and 7 in the same viewing window. (Use a viewing window in which $-2 \leq x \leq 1$ and $0 \leq y \leq 2$.) How do the graphs compare with each other? Which graph is on the top in the interval $(-\infty, 0)$? Which is on the bottom? Which graph is on the top in the interval $(0, \infty)$? Which is on the bottom? Repeat this experiment with the graphs of $y = b^x$ for $b = \frac{1}{3}, \frac{1}{5}$, and $\frac{1}{7}$. (Use a viewing window in which $-1 \leq x \leq 2$ and $0 \leq y \leq 2$.) What can you conclude about the shape of the graph of $y = b^x$ and the value of b?

In the following example, the graph of

$$y = a^x$$

is used to graph functions of the form

$$f(x) = b \pm a^{x+c}$$

where b and c are any real numbers.

Example 4 Library of Parent Functions: $f(x) = a^x$

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*, as shown in Figure 3.5.

b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *downward* two units, as shown in Figure 3.6.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x-axis, as shown in Figure 3.7.

d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y-axis, as shown in Figure 3.8.

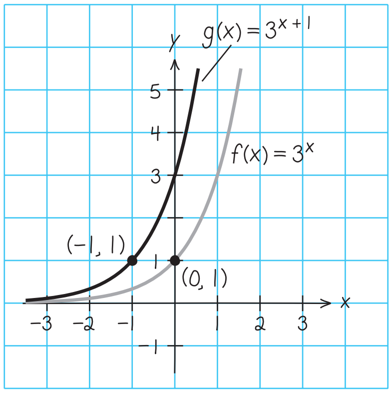


Figure 3.5

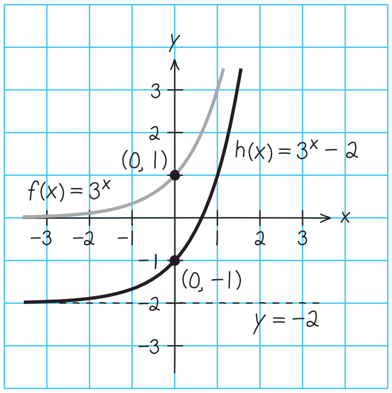


Figure 3.6

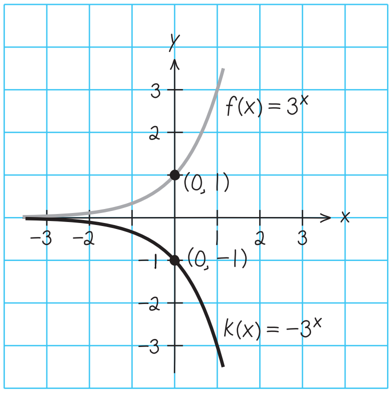


Figure 3.7

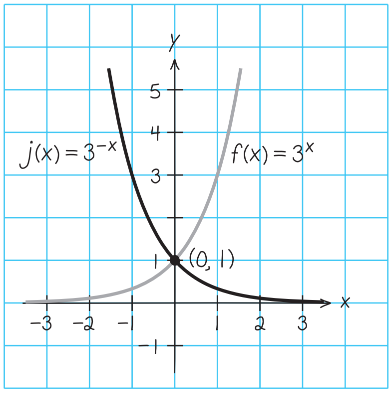


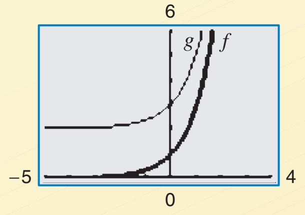
Figure 3.8

CHECKPOINT Now try Exercise 21.

Notice that the transformations in Figures 3.5, 3.7, and 3.8 keep the x-axis ($y = 0$) as a horizontal asymptote, but the transformation in Figure 3.6 yields a new horizontal asymptote of $y = -2$. Also, be sure to note how the y-intercept is affected by each transformation.

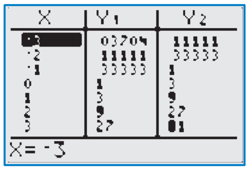
What's Wrong?

You use a graphing utility to graph $f(x) = 3^x$ and $g(x) = 3^{x+2}$, as shown in the figure. You use the graph to conclude that the graph of g can be obtained by shifting the graph of f upward two units. What's wrong?



Explore the Concept

The following table shows some points on the graphs in Figure 3.5. The functions $f(x)$ and $g(x)$ are represented by Y_1 and Y_2, respectively. Explain how you can use the table to describe the transformation.



The Natural Base e

For many applications, the convenient choice for a base is the irrational number

$$e = 2.718281828 \ldots.$$

This number is called the **natural base.** The function

$$f(x) = e^x$$

is called the **natural exponential function** and its graph is shown in Figure 3.9. The graph of the natural exponential function has the same basic characteristics as the graph of the function $f(x) = a^x$ (see page 182). Be sure you see that for the natural exponential function $f(x) = e^x$, e is the constant $2.718281828 \ldots$, whereas x is the variable.

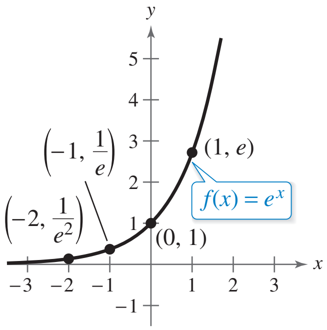


Figure 3.9 ***The Natural Exponential Function***

In Example 5, you will see that the number e can be approximated by the expression

$$\left(1 + \frac{1}{x}\right)^x \text{ for large values of } x.$$

Explore the Concept

Use your graphing utility to graph the functions

$$y_1 = 2^x$$
$$y_2 = e^x$$
$$y_3 = 3^x$$

in the same viewing window. From the relative positions of these graphs, make a guess as to the value of the real number e. Then try to find a number a such that the graphs of $y_2 = e^x$ and $y_4 = a^x$ are as close to each other as possible.

Example 5 Approximation of the Number e

Evaluate the expression

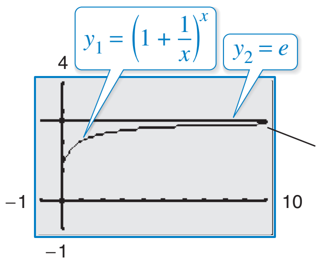
$$\left(1 + \frac{1}{x}\right)^x$$

for several large values of x to see that the values approach

$$e \approx 2.718281828$$

as x increases without bound.

Graphical Solution



As x increases, the graph of y_1 gets closer and closer to the graph of the line $y_2 = e$.

Figure 3.10

Numerical Solution

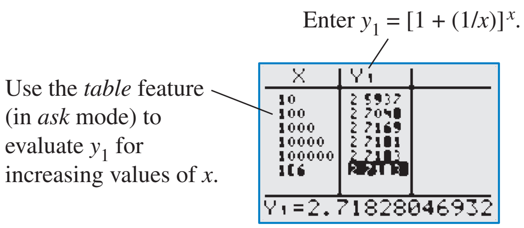


Figure 3.11

From Figure 3.11, it seems reasonable to conclude that

$$\left(1 + \frac{1}{x}\right)^x \to e \text{ as } x \to \infty.$$

CHECKPOINT Now try Exercise 27.

Example 6 Evaluating the Natural Exponential Function

Use a calculator to evaluate the function

$$f(x) = e^x$$

at each indicated value of x.

a. $x = -2$

b. $x = 0.25$

c. $x = -0.4$

d. $x = \frac{2}{3}$

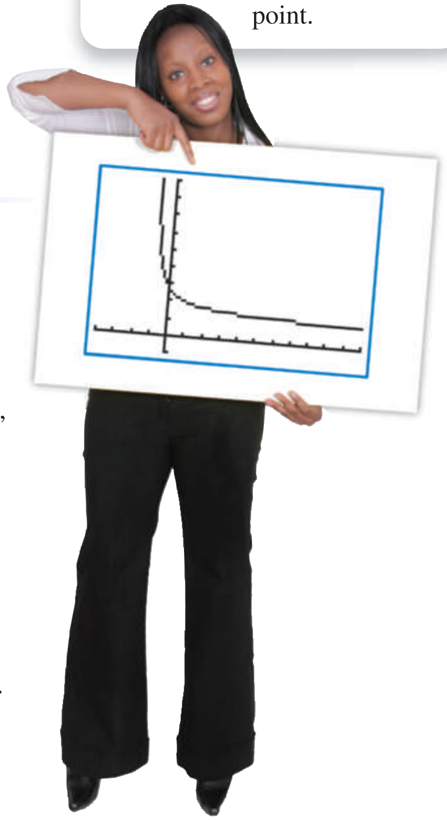
Solution

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(-2) = e^{-2}$	$\boxed{e^x}$ $\boxed{(-)}$ 2 $\boxed{\text{ENTER}}$	0.1353353
b. $f(0.25) = e^{0.25}$	$\boxed{e^x}$ .25 $\boxed{\text{ENTER}}$	1.2840254
c. $f(-0.4) = e^{-0.4}$	$\boxed{e^x}$ $\boxed{(-)}$ .4 $\boxed{\text{ENTER}}$	0.6703200
d. $f\left(\frac{2}{3}\right) = e^{2/3}$	$\boxed{e^x}$ $\boxed{(}$ 2 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\text{ENTER}}$	1.9477340

CHECKPOINT Now try Exercise 29.

Explore the Concept

Use a graphing utility to graph $y = (1 + x)^{1/x}$. Describe the behavior of the graph near $x = 0$. Is there a y-intercept? How does the behavior of the graph near $x = 0$ relate to the result of Example 5? Use the *table* feature of the graphing utility to create a table that shows values of y for values of x near $x = 0$ to help you describe the behavior of the graph near this point.



Example 7 Graphing Natural Exponential Functions

Sketch the graph of each natural exponential function.

a. $f(x) = 2e^{0.24x}$

b. $g(x) = \frac{1}{2}e^{-0.58x}$

Solution

To sketch these two graphs, you can use a calculator to construct a table of values, as shown below.

x	-3	-2	-1	0	1	2	3
$f(x)$	0.974	1.238	1.573	2.000	2.542	3.232	4.109
$g(x)$	2.849	1.595	0.893	0.500	0.280	0.157	0.088

After constructing the table, plot the points and connect them with smooth curves. Note that the graph in Figure 3.12 is increasing, whereas the graph in Figure 3.13 is decreasing. Use a graphing calculator to verify these graphs.

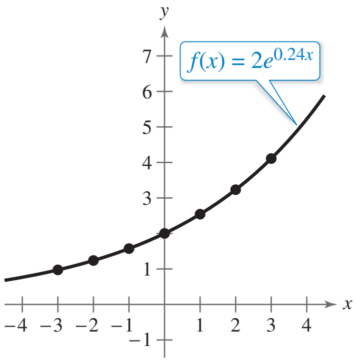


Figure 3.12

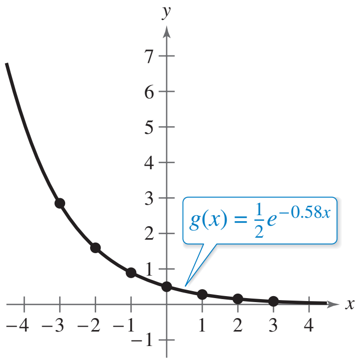


Figure 3.13

CHECKPOINT Now try Exercise 47.

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Applications

One of the most familiar examples of exponential growth is an investment earning *continuously compounded interest*. Suppose a principal P is invested at an annual interest rate r, compounded once a year. If the interest is added to the principal at the end of the year, then the new balance P_1 is

$$P_1 = P + Pr = P(1 + r).$$

This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown in the table.

Time in years	Balance after each compounding
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
$\vdots$	$\vdots$
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. (The product nt represents the total number of times the interest will be compounded.) Then the interest rate per compounding period is r/n, and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$ Amount (balance) with n compoundings per year

When the number of compoundings n increases without bound, the process approaches what is called **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

As m increases without bound, you know from Example 5 that

$$\left(1 + \frac{1}{m}\right)^{m}$$

approaches e. So, for continuous compounding, it follows that

$$P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \Rightarrow P[e]^{rt}$$

and you can write $A = Pe^{rt}$. This result is part of the reason that e is the "natural" choice for a base of an exponential function.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Explore the Concept

Use the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

to calculate the amount in an account when $P = \$3000$, $r = 6\%$, $t = 10$ years, and the interest is compounded (a) by the day, (b) by the hour, (c) by the minute, and (d) by the second. Does increasing the number of compoundings per year result in unlimited growth of the amount in the account? Explain.

Study Tip

The interest rate r in the formula for compound interest should be written as a decimal. For example, an interest rate of 7% would be written as $r = 0.07$.

Example 8 Finding the Balance for Compound Interest

A total of \$9000 is invested at an annual interest rate of 2.5%, compounded annually. Find the balance in the account after 5 years.

Algebraic Solution

In this case,

$$P = 9000,\ r = 2.5\% = 0.025,\ n = 1,\ t = 5.$$

Using the formula for compound interest with n compoundings per year, you have

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 9000\left(1 + \frac{0.025}{1}\right)^{1(5)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$= 9000(1.025)^5 \qquad \text{Simplify.}$$

$$\approx \$10{,}182.67. \qquad \text{Use a calculator.}$$

So, the balance in the account after 5 years will be about \$10,182.67.

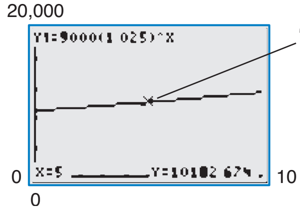
CHECKPOINT Now try Exercise 57.

Graphical Solution

Substitute the values for P, r, and n into the formula for compound interest with n compoundings per year and simplify to obtain

$$A = 9000(1.025)^t.$$

Use a graphing utility to graph $A = 9000(1.025)^t$. Then use the *value* feature to approximate the value of A when $t = 5$, as shown in Figure 3.14.



The balance in the account after 5 years will be about \$10,182.67.

Figure 3.14

Example 9 Finding Compound Interest

A total of \$12,000 is invested at an annual interest rate of 3%. Find the balance after 4 years for each type of compounding.

a. Quarterly

b. Continuous

Solution

a. For quarterly compoundings, $n = 4$. So, after 4 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 12{,}000\left(1 + \frac{0.03}{4}\right)^{4(4)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$\approx \$13{,}523.91. \qquad \text{Use a calculator.}$$

b. For continuous compounding, the balance is

$$A = Pe^{rt} \qquad \text{Formula for continuous compounding}$$

$$= 12{,}000e^{0.03(4)} \qquad \text{Substitute for } P, r, \text{ and } t.$$

$$\approx \$13{,}529.96. \qquad \text{Use a calculator.}$$

Note that a continuous-compounding account yields more than a quarterly-compounding account.

CHECKPOINT Now try Exercise 59.

Financial Analyst

Example 9 illustrates the following general rule. For a given principal, interest rate, and time, the more often the interest is compounded per year, the greater the balance will be. Moreover, the balance obtained by continuous compounding is greater than the balance obtained by compounding n times per year.

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Example 10 Radioactive Decay

Let y represent a mass, in grams, of radioactive strontium (^{90}Sr), whose half-life is 29 years. The quantity of strontium present after t years is

$$y = 10\left(\tfrac{1}{2}\right)^{t/29}.$$

a. What is the initial mass (when $t = 0$)?

b. How much of the initial mass is present after 80 years?

Algebraic Solution

a. $y = 10\left(\dfrac{1}{2}\right)^{t/29}$ Write original equation.

$= 10\left(\dfrac{1}{2}\right)^{0/29}$ Substitute 0 for t.

$= 10$ Simplify.

So, the initial mass is 10 grams.

b. $y = 10\left(\dfrac{1}{2}\right)^{t/29}$ Write original equation.

$= 10\left(\dfrac{1}{2}\right)^{80/29}$ Substitute 80 for t.

$\approx 10\left(\dfrac{1}{2}\right)^{2.759}$ Simplify.

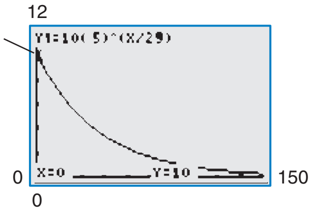
≈ 1.48 Use a calculator.

So, about 1.48 grams are present after 80 years.

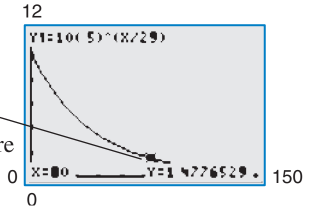
CHECKPOINT Now try Exercise 71.

Graphical Solution

a. When $t = 0$, $y = 10$. So, the initial mass is 10 grams.



b. When $t = 80$, $y \approx 1.48$. So, about 1.48 grams are present after 80 years.



Example 11 Population Growth

The approximate number of fruit flies in an experimental population after t hours is given by

$$Q(t) = 20e^{0.03t}$$

where $t \geq 0$.

a. Find the initial number of fruit flies in the population.

b. How large is the population of fruit flies after 72 hours?

c. Graph Q.

Solution

a. To find the initial population, evaluate $Q(t)$ when $t = 0$.

$$Q(0) = 20e^{0.03(0)} = 20e^0 = 20(1) = 20 \text{ flies}$$

b. After 72 hours, the population size is

$$Q(72) = 20e^{0.03(72)} = 20e^{2.16} \approx 173 \text{ flies.}$$

c. The graph of Q is shown in Figure 3.15.

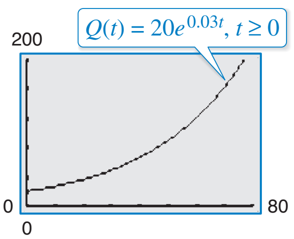


Figure 3.15

CHECKPOINT Now try Exercise 73.

3.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. Exponential and logarithmic functions are examples of nonalgebraic functions, also called _______ functions.

2. The exponential function $f(x) = e^x$ is called the _______ function, and the base e is called the _______ base.

3. What type of transformation of the graph of $f(x) = 5^x$ is the graph of $f(x + 1)$?

4. The formula $A = Pe^{rt}$ gives the balance A of an account earning what type of interest?

Procedures and Problem Solving

Evaluating Exponential Functions **In Exercises 5–8, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.**

Function	*Value*
5. $f(x) = 3.4^x$	$x = 6.8$
6. $f(x) = 1.2^x$	$x = \frac{1}{3}$
7. $g(x) = 5^x$	$x = -\pi$
8. $h(x) = 8.6^{-3x}$	$x = -\sqrt{2}$

Graphs of $y = a^x$ and $y = a^{-x}$ **In Exercises 9–16, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.**

9. $g(x) = 5^x$ **10.** $f(x) = \left(\frac{3}{2}\right)^x$

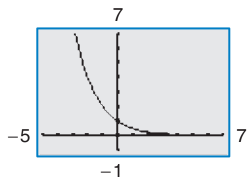
11. $f(x) = 5^{-x}$ **12.** $h(x) = \left(\frac{3}{2}\right)^{-x}$

13. $h(x) = 3^x$ **14.** $g(x) = 10^x$

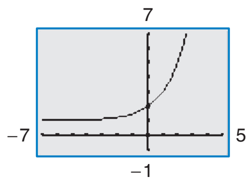
15. $g(x) = 3^{-x}$ **16.** $f(x) = 10^{-x}$

Library of Parent Functions **In Exercises 17–20, use the graph of $y = 2^x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

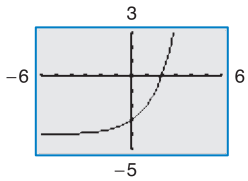
(a)



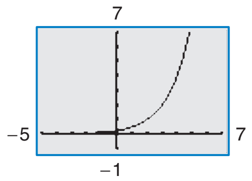
(b)



(c)



(d)



17. $f(x) = 2^{x-2}$ **18.** $f(x) = 2^{-x}$

19. $f(x) = 2^x - 4$ **20.** $f(x) = 2^x + 1$

Library of Parent Functions **In Exercises 21–26, use the graph of f to describe the transformation that yields the graph of g. Then sketch the graphs of f and g by hand.**

21. $f(x) = 3^x,\ g(x) = 3^{x-5}$

22. $f(x) = -2^x,\ g(x) = 5 - 2^x$

23. $f(x) = \left(\frac{3}{5}\right)^x,\ g(x) = -\left(\frac{3}{5}\right)^{x+4}$

24. $f(x) = 0.3^x,\ g(x) = -0.3^x + 5$

25. $f(x) = 4^x,\ g(x) = 4^{x-2} - 3$

26. $f(x) = \left(\frac{1}{2}\right)^x,\ g(x) = \left(\frac{1}{2}\right)^{-(x+4)}$

Approximation of a Power with Base e **In Exercises 27 and 28, show that the value of $f(x)$ approaches the value of $g(x)$ as x increases without bound (a) graphically and (b) numerically.**

27. $f(x) = [1 + (2/x)]^x,\quad g(x) = e^2$

28. $f(x) = [1 + (3/x)]^x,\quad g(x) = e^3$

Evaluating the Natural Exponential Function **In Exercises 29–32, use a calculator to evaluate the function at the indicated value of x. Round your result to the nearest thousandth.**

Function	*Value*
29. $f(x) = e^x$	$x = 9.2$
30. $f(x) = e^{-x}$	$x = -\frac{3}{4}$
31. $g(x) = 50e^{4x}$	$x = 0.02$
32. $h(x) = -5.5e^{-x}$	$x = 200$

Graphing an Exponential Function **In Exercises 33–48, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.**

33. $f(x) = \left(\frac{5}{2}\right)^x$ **34.** $f(x) = \left(\frac{5}{2}\right)^{-x}$

35. $f(x) = 6^x$ **36.** $f(x) = 2^{x-1}$

37. $f(x) = 3^{x+2}$ **38.** $y = 2^{-x^2}$

39. $y = 3^{x-2} + 1$ **40.** $y = 4^{x+1} - 2$

41. $f(x) = e^{-x}$ **42.** $s(t) = 3e^{-0.2t}$

43. $f(x) = 3e^{x+4}$ **44.** $f(x) = 2e^{-0.5x}$

45. $f(x) = 2 + e^{x-5}$ **46.** $g(x) = 2 - e^{-x}$

47. $s(t) = 2e^{0.12t}$ **48.** $g(x) = 1 + e^{-x}$

Finding Asymptotes **In Exercises 49–52, use a graphing utility to (a) graph the function and (b) find any asymptotes numerically by creating a table of values for the function.**

49. $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ **50.** $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

51. $f(x) = -\dfrac{6}{2 - e^{0.2x}}$ **52.** $f(x) = \dfrac{6}{2 - e^{0.2/x}}$

Finding Points of Intersection **In Exercises 53 and 54, use a graphing utility to find the point(s) of intersection, if any, of the graphs of the functions. Round your result to three decimal places.**

53. $y = 20e^{0.05x}$
$y = 1500$

54. $y = 100e^{0.01x}$
$y = 12{,}500$

Approximating Relative Extrema **In Exercises 55 and 56, (a) use a graphing utility to graph the function, (b) use the graph to find the open intervals on which the function is increasing and decreasing, and (c) approximate any relative maximum or minimum values.**

55. $f(x) = x^2e^{-x}$ **56.** $f(x) = 2x^2e^{x+1}$

Finding the Balance for Compound Interest **In Exercises 57–60, complete the table to determine the balance A for \$2500 invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous
A						

57. $r = 2\%, t = 10$ years **58.** $r = 6\%, t = 10$ years

59. $r = 4\%, t = 20$ years **60.** $r = 3\%, t = 40$ years

Finding the Balance for Compound Interest **In Exercises 61–64, complete the table to determine the balance A for \$12,000 invested at rate r for t years, compounded continuously.**

t	1	10	20	30	40	50
A						

61. $r = 4\%$ **62.** $r = 6\%$

63. $r = 3.5\%$ **64.** $r = 2.5\%$

Finding the Amount of an Annuity **In Exercises 65–68, you build an annuity by investing P dollars every month at interest rate r, compounded monthly. Find the amount A accrued after n months using the formula**

$$A = P\left[\frac{(1 + r/12)^n - 1}{r/12}\right]$$

where r is in decimal form.

65. $P = \$25, r = 0.12, n = 48$ months

66. $P = \$100, r = 0.09, n = 60$ months

67. $P = \$200, r = 0.06, n = 72$ months

68. $P = \$75, r = 0.03, n = 24$ months

69. MODELING DATA

There are three options for investing \$500. The first earns 7% compounded annually, the second earns 7% compounded quarterly, and the third earns 7% compounded continuously.

(a) Find equations that model the growth of each investment and use a graphing utility to graph each model in the same viewing window over a 20-year period.

(b) Use the graph from part (a) to determine which investment yields the highest return after 20 years. What are the differences in earnings among the three investments?

70. Radioactive Decay Let Q represent a mass, in grams, of radioactive radium (^{226}Ra), whose half-life is 1599 years. The quantity of radium present after t years is given by

$Q = 25\left(\frac{1}{2}\right)^{t/1599}.$

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 1000 years.

(c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 5000$.

(d) When will the quantity of radium be 0 grams? Explain.

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71. Radioactive Decay Let Q represent a mass, in grams, of carbon 14 (^{14}C), whose half-life is 5715 years. The quantity present after t years is given by $Q = 10\left(\frac{1}{2}\right)^{t/5715}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 2000 years.

(c) Sketch the graph of the function over the interval $t = 0$ to $t = 10{,}000$.

72. Algebraic-Graphical-Numerical Suppose the annual rate of inflation is 4% for the next 10 years. The approximate cost C of goods or services during these years is $C(t) = P(1.04)^t$, where t is the time (in years) and P is the present cost. An oil change for your car presently costs \$23.95. Use the following methods to approximate the cost 10 years from now.

(a) Use a graphing utility to graph the function and then use the *value* feature.

(b) Use the *table* feature of the graphing utility to find a numerical approximation.

(c) Use a calculator to evaluate the cost function algebraically.

73. Population Growth The projected populations of California for the years 2015 through 2030 can be modeled by $P = 34.706e^{0.0097t}$, where P is the population (in millions) and t is the time (in years), with $t = 15$ corresponding to 2015. (Source: U.S. Census Bureau)

(a) Use a graphing utility to graph the function for the years 2015 through 2030.

(b) Use the *table* feature of the graphing utility to create a table of values for the same time period as in part (a).

(c) According to the model, in what year will the population of California exceed 50 million?

74. *Why you should learn it* *(p. 180)* In early 2010, a new sedan had a manufacturer's suggested retail price of \$31,915. After t years, the sedan's value is given by

$$V(t) = 31{,}915\left(\tfrac{4}{5}\right)^t.$$

(a) Use a graphing utility to graph the function.

(b) Use the graphing utility to create a table of values that shows the value V for $t = 1$ to $t = 10$ years.

(c) According to the model, when will the sedan have no value?

Conclusions

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.**

75. $f(x) = 1^x$ is not an exponential function.

76. $e = \dfrac{271{,}801}{99{,}990}$

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bignecker 2010/used under license from Shutterstock.com

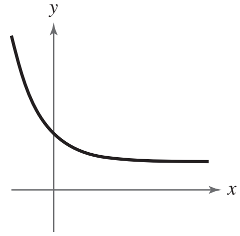
77. Library of Parent Functions Determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)

(a) $y = e^x + 1$

(b) $y = -e^{-x} + 1$

(c) $y = e^{-x} - 1$

(d) $y = e^{-x} + 1$



78. Exploration Use a graphing utility to graph $y_1 = e^x$ and each of the functions $y_2 = x^2$, $y_3 = x^3$, $y_4 = \sqrt{x}$, and $y_5 = |x|$ in the same viewing window.

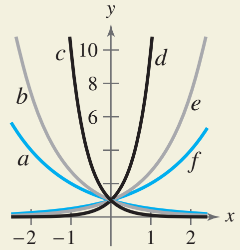
(a) Which function increases at the fastest rate for "large" values of x?

(b) Use the result of part (a) to make a conjecture about the rates of growth of $y_1 = e^x$ and $y = x^n$, where n is a natural number and x is "large."

(c) Use the results of parts (a) and (b) to describe what is implied when it is stated that a quantity is growing exponentially.

79. Think About It Graph $y = 3^x$ and $y = 4^x$. Use the graph to solve the inequality $3^x < 4^x$.

80. CAPSTONE The figure shows the graphs of $y = 2^x$, $y = e^x$, $y = 10^x$, $y = 2^{-x}$, $y = e^{-x}$, and $y = 10^{-x}$. Match each function with its graph. [The graphs are labeled (a) through (f).] Explain your reasoning.



Think About It **In Exercises 81–84, place the correct symbol (< or >) between the two of numbers.**

81. e^{π} ☐ π^e **82.** 2^{10} ☐ 10^2

83. 5^{-3} ☐ 3^{-5} **84.** $4^{1/2}$ ☐ $\left(\frac{1}{2}\right)^4$

Cumulative Mixed Review

Inverse Functions **In Exercises 85–88, determine whether the function has an inverse function. If it does, find f^{-1}.**

85. $f(x) = 5x - 7$ **86.** $f(x) = -\frac{2}{3}x + \frac{5}{2}$

87. $f(x) = \sqrt[3]{x + 8}$ **88.** $f(x) = \sqrt{x^2 + 6}$

89. *Make a Decision* To work an extended application analyzing the population per square mile in the United States, visit this textbook's *Companion Website*. (Data Source: U.S. Census Bureau)

3.2 Logarithmic Functions and Their Graphs

Logarithmic Functions

In Section 1.6, you studied the concept of an inverse function. There, you learned that when a function is one-to-one—that is, when the function has the property that no horizontal line intersects its graph more than once—the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 3.1, you will see that every function of the form

$$f(x) = a^x, \quad a > 0, a \neq 1$$

passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a*.**

Definition of Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

The function given by

$$f(x) = \log_a x$$ Read as "log base *a* of *x*."

is called the **logarithmic function with base *a*.**

From the definition above, you can see that every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form. The equations

$$y = \log_a x \quad \text{and} \quad x = a^y$$

are equivalent.

When evaluating logarithms, remember that *a logarithm is an exponent*. This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to get 8.

Example 1 Evaluating Logarithms

Use the definition of logarithmic function to evaluate each logarithm at the indicated value of x.

	Function	*Value*
a.	$f(x) = \log_2 x$	$x = 32$
b.	$f(x) = \log_3 x$	$x = 1$
c.	$f(x) = \log_4 x$	$x = 2$
d.	$f(x) = \log_{10} x$	$x = \frac{1}{100}$

Solution

a. $f(32) = \log_2 32 = 5$ because $2^5 = 32$.

b. $f(1) = \log_3 1 = 0$ because $3^0 = 1$.

c. $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.

d. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

CHECKPOINT Now try Exercise 23.

What you should learn

- Recognize and evaluate logarithmic functions with base *a*.
- Graph logarithmic functions with base *a*.
- Recognize, evaluate, and graph natural logarithmic functions.
- Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Logarithmic functions are useful in modeling data that represent quantities that increase or decrease slowly. For instance, Exercise 114 on page 201 shows how to use a logarithmic function to model the minimum required ventilation rates in public school classrooms.

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Study Tip

In this text, the parentheses in $\log_a(u)$ are sometimes omitted when u is an expression involving exponents, radicals, products, or quotients. For instance, $\log_{10}(2x)$ can be written as $\log_{10} 2x$. To evaluate $\log_{10} 2x$, find the logarithm of the product $2x$.

The logarithmic function with base 10 is called the **common logarithmic function.** On most calculators, this function is denoted by [LOG]. Example 2 shows how to use a calculator to evaluate common logarithmic functions. You will learn how to use a calculator to calculate logarithms to any base in the next section.

Example 2 Evaluating Common Logarithms on a Calculator

Use a calculator to evaluate the function

$$f(x) = \log_{10} x$$

at each value of x.

a. $x = 10$ **b.** $x = 2.5$

c. $x = -2$ **d.** $x = \frac{1}{4}$

Solution

Function Value	*Graphing Calculator Keystrokes*	*Display*
a. $f(10) = \log_{10} 10$	[LOG] 10 [ENTER]	1
b. $f(2.5) = \log_{10} 2.5$	[LOG] 2.5 [ENTER]	0.3979400
c. $f(-2) = \log_{10}(-2)$	[LOG] [(−)] 2 [ENTER]	ERROR
d. $f\left(\frac{1}{4}\right) = \log_{10} \frac{1}{4}$	[LOG] [(] 1 [÷] 4 [)] [ENTER]	-0.6020600

Note that the calculator displays an error message when you try to evaluate $\log_{10}(-2)$. In this case, there is no *real* power to which 10 can be raised to obtain -2.

CHECKPOINT Now try Exercise 27.

Technology Tip

Some graphing utilities do not give an error message for $\log_{10}(-2)$. Instead, the graphing utility will display a complex number. For the purpose of this text, however, it will be said that the domain of a logarithmic function is the set of positive *real* numbers.

The following properties follow directly from the definition of the logarithmic function with base a.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.

2. $\log_a a = 1$ because $a^1 = a$.

3. $\log_a a^x = x$ and $a^{\log_a x} = x$. Inverse Properties

4. If $\log_a x = \log_a y$, then $x = y$. One-to-One Property

Example 3 Using Properties of Logarithms

a. Solve for x: $\log_2 x = \log_2 3$

b. Solve for x: $\log_4 4 = x$

c. Simplify: $\log_5 5^x$

d. Simplify: $7^{\log_7 14}$

Solution

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.

b. Using Property 2, you can conclude that $x = 1$.

c. Using the Inverse Property (Property 3), it follows that $\log_5 5^x = x$.

d. Using the Inverse Property (Property 3), it follows that $7^{\log_7 14} = 14$.

CHECKPOINT Now try Exercise 31.

Graphs of Logarithmic Functions

To sketch the graph of

$$y = \log_a x$$

you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 4 Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

Solution

a. For $f(x) = 2^x$, construct a table of values. By plotting these points and connecting them with a smooth curve, you obtain the graph of f shown in Figure 3.16.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

b. Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 3.16.

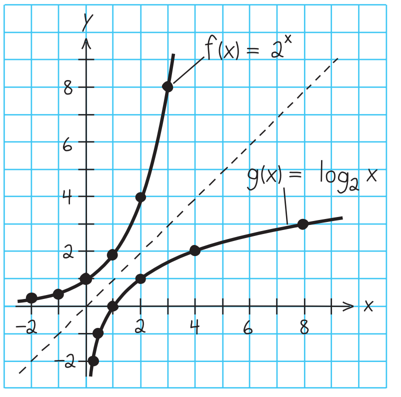


Figure 3.16

CHECKPOINT Now try Exercise 41.

Before you can confirm the result of Example 4 using a graphing utility, you need to know how to enter $\log_2 x$. You will learn how to do this using the *change-of-base formula* discussed in Section 3.3.

Example 5 Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function $f(x) = \log_{10} x$ by hand.

Solution

Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the Inverse Property of Logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 3.17.

	Without calculator				With calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log_{10} x$	-2	-1	0	1	0.301	0.699	0.903

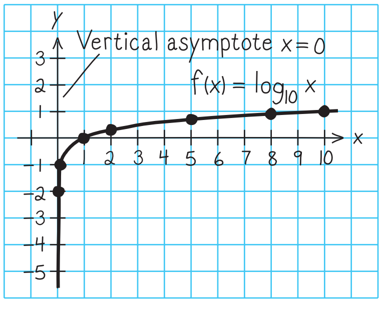


Figure 3.17

CHECKPOINT Now try Exercise 45.

The nature of the graph in Figure 3.17 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$.

Library of Parent Functions: Logarithmic Function

The *parent logarithmic function*

$$f(x) = \log_a x, \quad a > 0, \ a \neq 1$$

is the inverse function of the exponential function. Its domain is the set of positive real numbers and its range is the set of all real numbers. This is the opposite of the exponential function. Moreover, the logarithmic function has the y-axis as a vertical asymptote, whereas the exponential function has the x-axis as a horizontal asymptote. Many real-life phenomena with slow rates of growth can be modeled by logarithmic functions. The basic characteristics of the logarithmic function are summarized below and on the inside cover of this text.

Graph of $f(x) = \log_a x,\ a > 1$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Intercept: $(1, 0)$

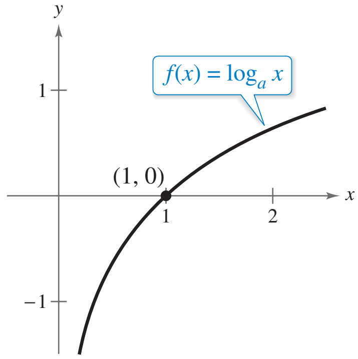
Increasing on $(0, \infty)$

y-axis is a vertical asymptote

($\log_a x \to -\infty$ as $x \to 0^+$)

Continuous

Reflection of graph of $f(x) = a^x$ in the line $y = x$



Explore the Concept

Use a graphing utility to graph $y = \log_{10} x$ and $y = 8$ in the same viewing window. Find a viewing window that shows the point of intersection. What is the point of intersection? Use the point of intersection to complete the equation $\log_{10} \square = 8$.

Example 6 Library of Parent Functions $f(x) = \log_a x$

Each of the following functions is a transformation of the graph of

$$f(x) = \log_{10} x.$$

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *right*, as shown in Figure 3.18.

b. Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units *upward*, as shown in Figure 3.19.

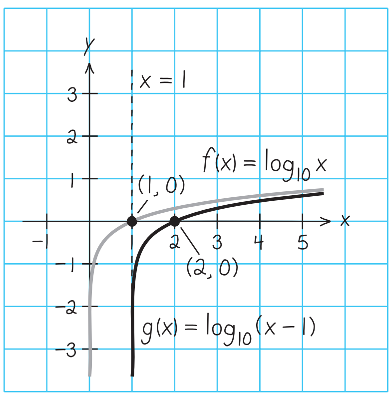


Figure 3.18

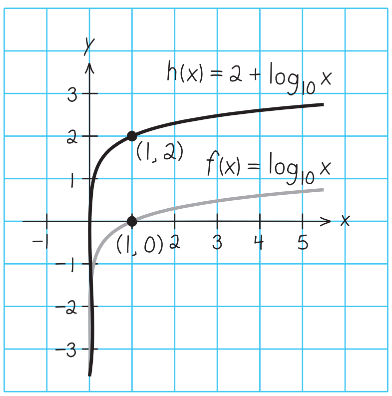


Figure 3.19

Notice that the transformation in Figure 3.19 keeps the y-axis as a vertical asymptote, but the transformation in Figure 3.18 yields the new vertical asymptote $x = 1$.

CHECKPOINT Now try Exercise 55.

The Natural Logarithmic Function

By looking back at the graph of the natural exponential function introduced in Section 3.1, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as "the natural log of x" or "el en of x."

The Natural Logarithmic Function

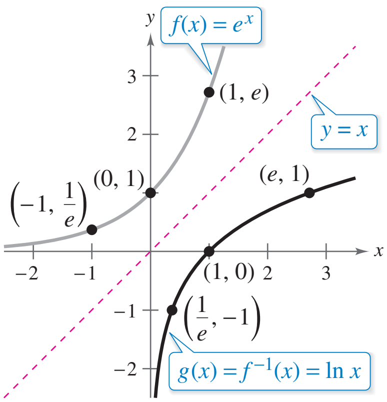
For $x > 0$,

$$y = \ln x \quad \text{if and only if} \quad x = e^y.$$

The function given by

$$f(x) = \log_e x = \ln x$$

is called the **natural logarithmic function.**



Reflection of graph of $f(x) = e^x$ in the line $y = x$

Figure 3.20

The equations $y = \ln x$ and $x = e^y$ are equivalent. Note that the natural logarithm $\ln x$ is written without a base. The base is understood to be e.

Because the functions

$$f(x) = e^x \quad \text{and} \quad g(x) = \ln x$$

are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 3.20.

Example 7 Evaluating the Natural Logarithmic Function

Use a calculator to evaluate the function

$$f(x) = \ln x$$

at each indicated value of x.

a. $x = 2$

b. $x = 0.3$

c. $x = -1$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(2) = \ln 2$	LN 2 ENTER	0.6931472
b. $f(0.3) = \ln 0.3$	LN .3 ENTER	−1.2039728
c. $f(-1) = \ln(-1)$	LN (−) 1 ENTER	ERROR

CHECKPOINT Now try Exercise 77.

Technology Tip

On most calculators, the natural logarithm is denoted by LN, as illustrated in Example 7.

Study Tip

In Example 7(c), be sure you see that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of $\ln x$ is the set of *positive* real numbers (see Figure 3.20). So, $\ln(-1)$ is undefined.

The four properties of logarithms listed on page 193 are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$. Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. One-to-One Property

Example 8 Using Properties of Natural Logarithms

Use the properties of natural logarithms to rewrite each expression.

a. $\ln \frac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $4 \ln 1$ **d.** $2 \ln e$

Solution

a. $\ln \frac{1}{e} = \ln e^{-1} = -1$ Inverse Property

b. $e^{\ln 5} = 5$ Inverse Property

c. $4 \ln 1 = 4(0) = 0$ Property 1

d. $2 \ln e = 2(1) = 2$ Property 2

CHECKPOINT Now try Exercise 81.

Example 9 Finding the Domains of Logarithmic Functions

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

Algebraic Solution

a. Because $\ln(x - 2)$ is defined only when

$$x - 2 > 0$$

it follows that the domain of f is $(2, \infty)$.

b. Because $\ln(2 - x)$ is defined only when

$$2 - x > 0$$

it follows that the domain of g is $(-\infty, 2)$.

c. Because $\ln x^2$ is defined only when

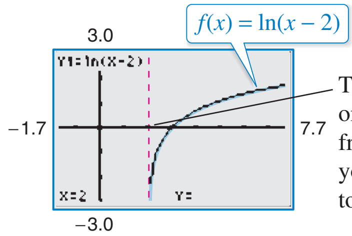
$$x^2 > 0$$

it follows that the domain of h is all real numbers except $x = 0$.

CHECKPOINT Now try Exercise 89.

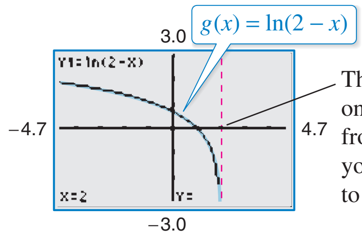
Graphical Solution

a.



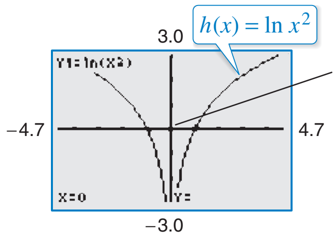
The x-coordinates of points on the graph appear to extend from the right of 2 to ∞. So, you can estimate the domain to be $(2, \infty)$.

b.



The x-coordinates of points on the graph appear to extend from $-\infty$ to the left of 2. So, you can estimate the domain to be $(-\infty, 2)$.

c.



The x-coordinates of points on the graph appear to include all real numbers except 0. So, you can estimate the domain to be all real numbers except $x = 0$.

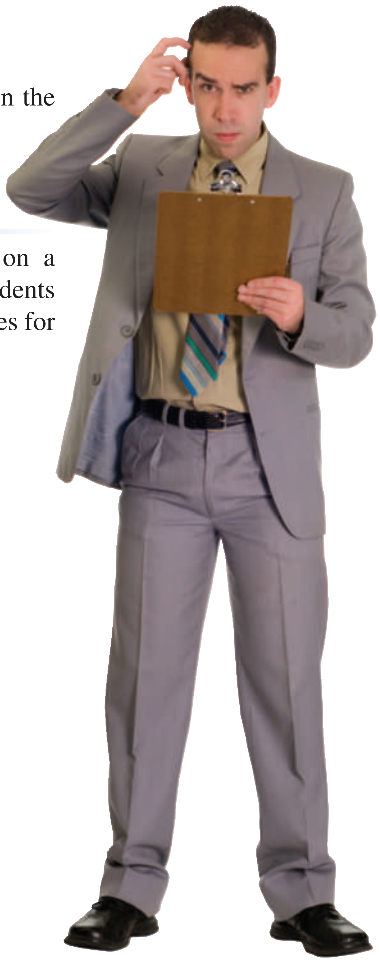
In Example 9, suppose you had been asked to analyze the function $h(x) = \ln|x - 2|$. How would the domain of this function compare with the domains of the functions given in parts (a) and (b) of the example?

Technology Tip

When a graphing utility graphs a logarithmic function, it may appear that the graph has an endpoint. This is because some graphing utilities have a limited resolution. So, in this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.

Application

Logarithmic functions are used to model many situations in real life, as shown in the next example.



Psychologist

Example 10 Psychology

Students participating in a psychology experiment attended several lectures on a subject and were given an exam. Every month for a year after the exam, the students were retested to see how much of the material they remembered. The average scores for the group are given by the *human memory model*

$$f(t) = 75 - 6\ln(t + 1), \quad 0 \le t \le 12$$

where t is the time in months. The graph of f is shown in Figure 3.21.

a. What was the average score on the original exam $(t = 0)$?

b. What was the average score at the end of $t = 2$ months?

c. What was the average score at the end of $t = 6$ months?

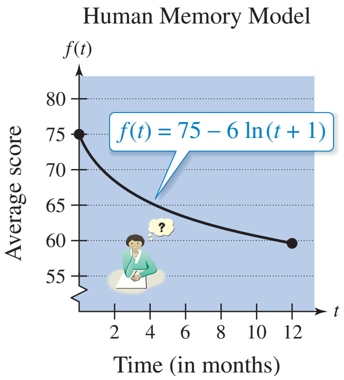


Figure 3.21

Algebraic Solution

a. The original average score was

$$\begin{aligned} f(0) &= 75 - 6\ln(0 + 1) \\ &= 75 - 6\ln 1 \\ &= 75 - 6(0) \\ &= 75. \end{aligned}$$

b. After 2 months, the average score was

$$\begin{aligned} f(2) &= 75 - 6\ln(2 + 1) \\ &= 75 - 6\ln 3 \\ &\approx 75 - 6(1.0986) \\ &\approx 68.41. \end{aligned}$$

c. After 6 months, the average score was

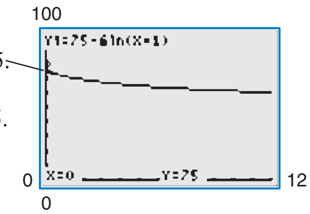
$$\begin{aligned} f(6) &= 75 - 6\ln(6 + 1) \\ &= 75 - 6\ln 7 \\ &\approx 75 - 6(1.9459) \\ &\approx 63.32. \end{aligned}$$

CHECKPOINT Now try Exercise 109.

Graphical Solution

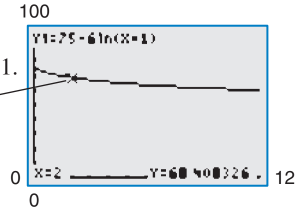
a.

When $t = 0$, $f(0) = 75$. So, the original average score was 75.



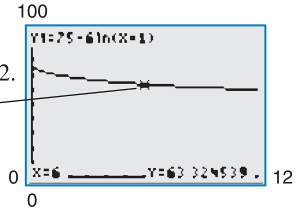
b.

When $t = 2$, $f(2) \approx 68.41$. So, the average score after 2 months was about 68.41.



c.

When $t = 6$, $f(6) \approx 63.32$. So, the average score after 6 months was about 63.32.



3.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. The inverse function of the exponential function $f(x) = a^x$ is called the _______ with base a.

2. The base of the _______ logarithmic function is 10, and the base of the _______ logarithmic function is e.

3. The inverse properties of logarithms are $\log_a a^x = x$ and _______ .

4. If $x = e^y$, then $y =$ _______.

5. What exponential equation is equivalent to the logarithmic equation $\log_a b = c$?

6. For what value(s) of x is $\ln x = \ln 7$?

Procedures and Problem Solving

Rewriting Logarithmic Equations **In Exercises 7–14, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 25 = 2$ is $5^2 = 25$.**

7. $\log_4 64 = 3$ **8.** $\log_3 81 = 4$

9. $\log_7 \frac{1}{49} = -2$ **10.** $\log_{10} \frac{1}{1000} = -3$

11. $\log_{32} 4 = \frac{2}{5}$ **12.** $\log_{16} 8 = \frac{3}{4}$

13. $\log_2 \sqrt{2} = \frac{1}{2}$ **14.** $\log_5 \sqrt[3]{25} = \frac{2}{3}$

Rewriting Exponential Equations **In Exercises 15–22, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.**

15. $5^3 = 125$ **16.** $8^2 = 64$

17. $81^{1/4} = 3$ **18.** $9^{3/2} = 27$

19. $6^{-2} = \frac{1}{36}$ **20.** $10^{-3} = 0.001$

21. $g^a = 4$ **22.** $n^t = 10$

Evaluating Logarithms **In Exercises 23–26, use the definition of logarithmic function to evaluate the function at the indicated value of x without using a calculator.**

	Function	*Value*
✓ **23.**	$f(x) = \log_2 x$	$x = 16$
24.	$f(x) = \log_{16} x$	$x = \frac{1}{4}$
25.	$g(x) = \log_{10} x$	$x = \frac{1}{1000}$
26.	$g(x) = \log_{10} x$	$x = 10{,}000$

Evaluating Common Logarithms on a Calculator **In Exercises 27–30, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.**

	Function	*Value*
✓ **27.**	$f(x) = \log_{10} x$	$x = 345$
28.	$f(x) = \log_{10} x$	$x = \frac{4}{5}$
29.	$h(x) = 6 \log_{10} x$	$x = 14.8$
30.	$h(x) = 1.9 \log_{10} x$	$x = 4.3$

Using Properties of Logarithms **In Exercises 31–36, solve the equation for x.**

✓ **31.** $\log_7 x = \log_7 9$ **32.** $\log_5 5 = x$

33. $\log_4 4^2 = x$ **34.** $\log_3 3^{-5} = x$

35. $\log_8 x = \log_8 10^{-1}$ **36.** $\log_4 4^3 = x$

Using Properties of Logarithms **In Exercises 37–40, use the properties of logarithms to simplify the expression.**

37. $\log_4 4^{3x}$ **38.** $6^{\log_6 36}$

39. $3 \log_2 \frac{1}{2}$ **40.** $\frac{1}{4} \log_4 16$

Graphs of Exponential and Logarithmic Functions **In Exercises 41–44, sketch the graph of f. Then use the graph of f to sketch the graph of g.**

✓ **41.** $f(x) = 3^x$, $g(x) = \log_3 x$ **42.** $f(x) = 5^x$, $g(x) = \log_5 x$

43. $f(x) = 15^x$, $g(x) = \log_{15} x$ **44.** $f(x) = 4^x$, $g(x) = \log_4 x$

Sketching the Graph of a Logarithmic Function **In Exercises 45–50, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand.**

✓ **45.** $y = \log_2(x + 2)$ **46.** $y = \log_2(x - 1)$

47. $y = 1 + \log_2 x$

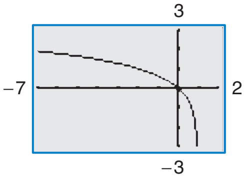
48. $y = 2 - \log_2 x$

49. $y = 1 + \log_2(x - 2)$

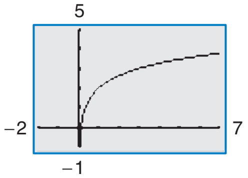
50. $y = 2 + \log_2(x + 1)$

Library of Parent Functions **In Exercises 51–54, use the graph of $y = \log_3 x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

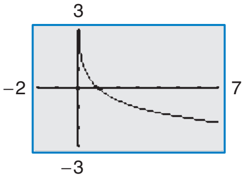
(a)



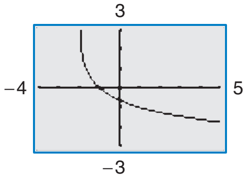
(b)



(c)



(d)



51. $f(x) = \log_3 x + 2$

52. $f(x) = -\log_3 x$

53. $f(x) = -\log_3(x + 2)$

54. $f(x) = \log_3(1 - x)$

Library of Parent Functions **In Exercises 55–60, describe the transformation of the graph of f that yields the graph of g.**

55. $f(x) = \log_{10} x,\quad g(x) = -\log_{10} x$

56. $f(x) = \log_{10} x,\quad g(x) = \log_{10}(x + 7)$

57. $f(x) = \log_2 x,\quad g(x) = 4 - \log_2 x$

58. $f(x) = \log_2 x,\quad g(x) = 3 + \log_2 x$

59. $f(x) = \log_8 x,\quad g(x) = -2 + \log_8(x + 3)$

60. $f(x) = \log_8 x,\quad g(x) = 4 + \log_8(x - 1)$

Rewriting Logarithmic Equations **In Exercises 61–68, write the logarithmic equation in exponential form. For example, the exponential form of $\ln 5 = 1.6094\ldots$ is $e^{1.6094\ldots} = 5$.**

61. $\ln 1 = 0$ **62.** $\ln 4 = 1.3862\ldots$

63. $\ln e = 1$ **64.** $\ln e^3 = 3$

65. $\ln \sqrt{e} = \dfrac{1}{2}$ **66.** $\ln \dfrac{1}{e^2} = -2$

67. $\ln 9 = 2.1972\ldots$ **68.** $\ln \sqrt[3]{e} = \frac{1}{3}$

Rewriting Exponential Equations **In Exercises 69–76, write the exponential equation in logarithmic form. For example, the logarithmic form of $e^2 = 7.3890\ldots$ is $\ln 7.3890\ldots = 2$.**

69. $e^3 = 20.0855\ldots$ **70.** $e^4 = 54.5981\ldots$

71. $e^{1.3} = 3.6692\ldots$ **72.** $e^{2.5} = 12.1824\ldots$

73. $\sqrt[3]{e} = 1.3956\ldots$ **74.** $\dfrac{1}{e^4} = 0.0183\ldots$

75. $\sqrt{e^3} = 4.4816\ldots$ **76.** $e^{3/4} = 2.1170\ldots$

Evaluating the Natural Logarithmic Function **In Exercises 77–80, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.**

Function	*Value*
77. $f(x) = \ln x$	$x = \sqrt{42}$
78. $f(x) = \ln x$	$x = 18.31$
79. $f(x) = -\ln x$	$x = \frac{1}{2}$
80. $f(x) = 3 \ln x$	$x = 0.75$

Using Properties of Natural Logarithms **In Exercises 81–88, use the properties of natural logarithms to rewrite the expression.**

81. $\ln e^2$ **82.** $-\ln e$

83. $e^{\ln 1.8}$ **84.** $7 \ln e^0$

85. $e \ln 1$ **86.** $e^{\ln 22}$

87. $\ln e^{\ln e}$ **88.** $\ln \dfrac{1}{e^4}$

Library of Parent Functions **In Exercises 89–92, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand. Verify using a graphing utility.**

89. $f(x) = \ln(x - 1)$ **90.** $h(x) = \ln(x + 1)$

91. $g(x) = \ln(-x)$ **92.** $f(x) = \ln(3 - x)$

Library of Parent Functions **In Exercises 93–98, use the graph of $f(x) = \ln x$ to describe the transformation that yields the graph of g.**

93. $g(x) = \ln(x + 3)$ **94.** $g(x) = \ln(x - 4)$

95. $g(x) = \ln x - 5$ **96.** $g(x) = \ln x + 4$

97. $g(x) = \ln(x - 1) + 2$ **98.** $g(x) = \ln(x + 2) - 5$

Analyzing Graphs of Functions **In Exercises 99–108, (a) use a graphing utility to graph the function, (b) find the domain, (c) use the graph to find the open intervals on which the function is increasing and decreasing, and (d) approximate any relative maximum or minimum values of the function. Round your result to three decimal places.**

99. $f(x) = \dfrac{x}{2} - \ln \dfrac{x}{4}$ **100.** $g(x) = \dfrac{12 \ln x}{x}$

101. $h(x) = 4x \ln x$ **102.** $f(x) = \dfrac{x}{\ln x}$

103. $f(x) = \ln \dfrac{x + 2}{x - 1}$ **104.** $f(x) = \ln \dfrac{2x}{x + 2}$

105. $f(x) = \ln \dfrac{x^2}{10}$ **106.** $f(x) = \ln \dfrac{x}{x^2 + 1}$

107. $f(x) = \sqrt{\ln x}$ **108.** $f(x) = (\ln x)^2$

109. Psychology Students in a mathematics class were given an exam and then tested monthly with an equivalent exam. The average scores for the class are given by the human memory model

$$f(t) = 80 - 17\log_{10}(t + 1), \quad 0 \le t \le 12$$

where t is the time in months.

(a) What was the average score on the original exam $(t = 0)$?

(b) What was the average score after 4 months?

(c) What was the average score after 10 months?

Verify your answers in parts (a), (b), and (c) using a graphing utility.

110. MODELING DATA

The table shows the temperatures T (in degrees Fahrenheit) at which water boils at selected pressures p (in pounds per square inch). (Source: Standard Handbook of Mechanical Engineers)

Pressure, p	Temperature, T
5	162.24°
10	193.21°
14.696 (1 atm)	212.00°
20	227.96°
30	250.33°
40	267.25°
60	292.71°
80	312.03°
100	327.81°

A model that approximates the data is

$$T = 87.97 + 34.96\ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the graph to estimate the pressure at which the boiling point of water is 300°F.

(c) Calculate T when the pressure is 74 pounds per square inch. Verify your answer graphically.

111. Finance A principal P, invested at $5\frac{1}{2}\%$ and compounded continuously, increases to an amount K times the original principal after t years, where $t = (\ln K)/0.055$.

(a) Complete the table and interpret your results.

K	1	2	4	6	8	10	12
t							

(b) Use a graphing utility to graph the function.

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112. Science The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10\log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Determine the number of decibels of a sound with an intensity of 1 watt per square meter.

(b) Determine the number of decibels of a sound with an intensity of 10^{-2} watt per square meter.

(c) The intensity of the sound in part (a) is 100 times as great as that in part (b). Is the number of decibels 100 times as great? Explain.

113. Real Estate The model

$$t = 16.625\ln\frac{x}{x - 750}, \quad x > 750$$

approximates the length of a home mortgage of \$150,000 at 6% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.

(a) Use the model to approximate the lengths of a \$150,000 mortgage at 6% when the monthly payment is \$897.72 and when the monthly payment is \$1659.24.

(b) Approximate the total amounts paid over the term of the mortgage with a monthly payment of \$897.72 and with a monthly payment of \$1659.24. What amount of the total is interest costs for each payment?

114. *Why you should learn it* *(p. 192)* The rate of ventilation required in a public school classroom depends on the volume of air space per child. The model

$$y = 80.4 - 11\ln x, \quad 100 \le x \le 1500$$

approximates the minimum required rate of ventilation y (in cubic feet per minute per child) in a classroom with x cubic feet of air space per child.

(a) Use a graphing utility to graph the function and approximate the required rate of ventilation in a room with 300 cubic feet of air space per child.

(b) A classroom of 30 students has an air conditioning system that moves 450 cubic feet of air per minute. Determine the rate of ventilation per child.

(c) Use the graph in part (a) to estimate the minimum required air space per child for the classroom in part (b).

(d) The classroom in part (b) has 960 square feet of floor space and a ceiling that is 12 feet high. Is the rate of ventilation for this classroom adequate? Explain.

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Conclusions

True or False? **In Exercises 115 and 116, determine whether the statement is true or false. Justify your answer.**

115. You can determine the graph of $f(x) = \log_6 x$ by graphing $g(x) = 6^x$ and reflecting it about the x-axis.

116. The graph of $f(x) = \log_3 x$ contains the point $(27, 3)$.

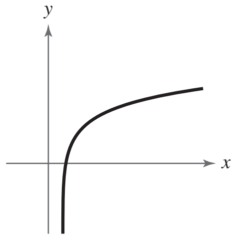
Think About It **In Exercises 117–120, find the value of the base b so that the graph of $f(x) = \log_b x$ contains the given point.**

117. $(32, 5)$ **118.** $(81, 4)$

119. $\left(\frac{1}{16}, 2\right)$ **120.** $\left(\frac{1}{27}, 3\right)$

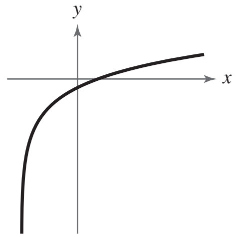
Library of Parent Functions **In Exercises 121 and 122, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)**

121.



(a) $y = \log_2(x + 1) + 2$
(b) $y = \log_2(x - 1) + 2$
(c) $y = 2 - \log_2(x - 1)$
(d) $y = \log_2(x + 2) + 1$

122.



(a) $y = \ln(x - 1) + 2$
(b) $y = \ln(x + 2) - 1$
(c) $y = 2 - \ln(x - 1)$
(d) $y = \ln(x - 2) + 1$

123. Writing Explain why $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

124. Exploration Let $f(x) = \ln x$ and $g(x) = x^{1/n}$.

(a) Use a graphing utility to graph g (for $n = 2$) and f in the same viewing window.

(b) Determine which function is increasing at a greater rate as x approaches infinity.

(c) Repeat parts (a) and (b) for $n = 3, 4,$ and 5. What do you notice?

125. Exploration

(a) Use a graphing utility to compare the graph of the function $y = \ln x$ with the graph of each function.

$y_1 = x - 1,\ y_2 = (x - 1) - \frac{1}{2}(x - 1)^2,$

$y_3 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$

(b) Identify the pattern of successive polynomials given in part (a). Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = \ln x$. What do you think the pattern implies?

126. CAPSTONE The following table of values was obtained by evaluating a function. Determine which of the statements may be true and which must be false. Explain your reasoning.

x	1	2	8
y	0	1	3

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

127. Exploration

(a) Use a graphing utility to complete the table for the function

$$f(x) = \frac{\ln x}{x}.$$

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(b) Use the table in part (a) to determine what value $f(x)$ approaches as x increases without bound. Use the graphing utility to confirm your result.

128. Writing Use a graphing utility to determine how many months it would take for the average score in Example 10 to decrease to 60. Explain your method of solving the problem. Describe another way that you can use the graphing utility to determine the answer. Also, based on the shape of the graph, does the rate at which a student forgets information *increase* or *decrease* with time? Explain.

Cumulative Mixed Review

Factoring a Polynomial **In Exercises 129–136, factor the polynomial.**

129. $x^2 + 2x - 3$ **130.** $2x^2 + 3x - 5$

131. $12x^2 + 5x - 3$ **132.** $16x^2 + 16x + 7$

133. $16x^2 - 25$ **134.** $36x^2 - 49$

135. $2x^3 + x^2 - 45x$ **136.** $3x^3 - 5x^2 - 12x$

Evaluating an Arithmetic Combination of Functions **In Exercises 137 and 138, evaluate the function for $f(x) = 3x + 2$ and $g(x) = x^3 - 1$.**

137. $(f + g)(2)$ **138.** $(f - g)(-1)$

Using Graphs **In Exercises 139–142, solve the equation graphically.**

139. $5x - 7 = x + 4$ **140.** $-2x + 3 = 8x$

141. $\sqrt{3x - 2} = 9$ **142.** $\sqrt{x - 11} = x + 2$

3.3 Properties of Logarithms

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms to other bases. To do this, you can use the following **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b. The constant multiplier is

$$\frac{1}{\log_b a}.$$

Example 1 Changing Bases Using Common Logarithms

a. $\log_4 25 = \dfrac{\log_{10} 25}{\log_{10} 4}$ $\qquad \log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

$\approx \dfrac{1.39794}{0.60206}$ $\qquad$ Use a calculator.

≈ 2.32 $\qquad$ Simplify.

b. $\log_2 12 = \dfrac{\log_{10} 12}{\log_{10} 2} \approx \dfrac{1.07918}{0.30103} \approx 3.58$

CHECKPOINT Now try Exercise 13.

Example 2 Changing Bases Using Natural Logarithms

a. $\log_4 25 = \dfrac{\ln 25}{\ln 4}$ $\qquad \log_a x = \dfrac{\ln x}{\ln a}$

$\approx \dfrac{3.21888}{1.38629}$ $\qquad$ Use a calculator.

≈ 2.32 $\qquad$ Simplify.

b. $\log_2 12 = \dfrac{\ln 12}{\ln 2} \approx \dfrac{2.48491}{0.69315} \approx 3.58$

CHECKPOINT Now try Exercise 19.

Notice in Examples 1 and 2 that the result is the same whether common logarithms or natural logarithms are used in the change-of-base formula.

What you should learn

- Rewrite logarithms with different bases.
- Use properties of logarithms to evaluate or rewrite logarithmic expressions.
- Use properties of logarithms to expand or condense logarithmic expressions.
- Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Logarithmic functions can be used to model and solve real-life problems, such as the model for the number of decibels of a sound in Exercise 107 on page 208.

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Properties of Logarithms

You know from the previous section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents (see Section 3.1) should have corresponding properties involving logarithms. For instance, the exponential property

$$a^0 = 1$$

has the corresponding logarithmic property

$$\log_a 1 = 0.$$

Properties of Logarithms (See the proof on page 251.)

Let a be a positive real number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	*Natural Logarithm*
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Study Tip

There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is *not* equal to $\log_a x + \log_a y$.

Example 3 Using Properties of Logarithms

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$

b. $\ln \frac{2}{27}$

Solution

a. $\ln 6 = \ln(2 \cdot 3)$ — Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ — Product Property

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ — Quotient Property

$= \ln 2 - \ln 3^3$ — Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ — Power Property

CHECKPOINT Now try Exercise 21.

Example 4 Using Properties of Logarithms

Use the properties of logarithms to verify that $-\log_{10} \frac{1}{100} = \log_{10} 100$.

Solution

$-\log_{10} \frac{1}{100} = -\log_{10}(100^{-1})$ — Rewrite $\frac{1}{100}$ as 100^{-1}.

$= -(-1) \log_{10} 100$ — Power Property

$= \log_{10} 100$ — Simplify.

CHECKPOINT Now try Exercise 45.

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because they convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

Example 5 Expanding Logarithmic Expressions

Use the properties of logarithms to expand each expression.

a. $\log_4 5x^3y$

b. $\ln \dfrac{\sqrt{3x-5}}{7}$

Solution

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Product Property

$= \log_4 5 + 3\log_4 x + \log_4 y$ Power Property

b. $\ln \dfrac{\sqrt{3x-5}}{7} = \ln \dfrac{(3x-5)^{1/2}}{7}$ Rewrite radical using rational exponent.

$= \ln(3x-5)^{1/2} - \ln 7$ Quotient Property

$= \dfrac{1}{2}\ln(3x-5) - \ln 7$ Power Property

CHECKPOINT Now try Exercise 63.

Explore the Concept

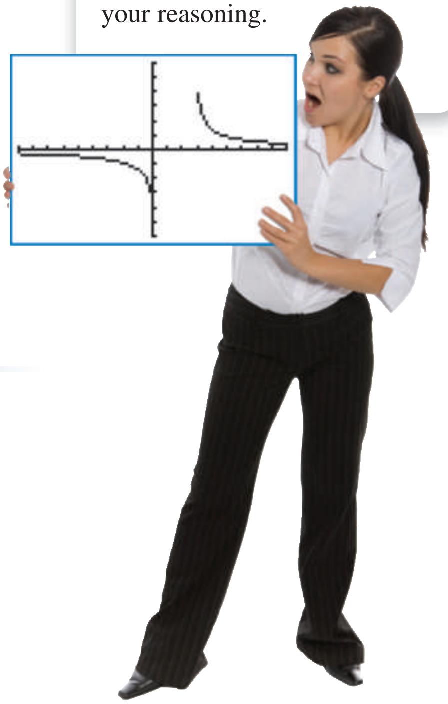
Use a graphing utility to graph the functions

$$y = \ln x - \ln(x-3)$$

and

$$y = \ln \frac{x}{x-3}$$

in the same viewing window. Does the graphing utility show the functions with the same domain? Should it? Explain your reasoning.



In Example 5, the properties of logarithms were used to *expand* logarithmic expressions. In Example 6, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

Example 6 Condensing Logarithmic Expressions

Use the properties of logarithms to condense each expression.

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1)$

b. $2\ln(x+2) - \ln x$

c. $\frac{1}{3}[\log_2 x + \log_2(x-4)]$

Solution

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1) = \log_{10} x^{1/2} + \log_{10}(x+1)^3$ Power Property

$= \log_{10}\left[\sqrt{x}(x+1)^3\right]$ Product Property

b. $2\ln(x+2) - \ln x = \ln(x+2)^2 - \ln x$ Power Property

$= \ln \dfrac{(x+2)^2}{x}$ Quotient Property

c. $\frac{1}{3}[\log_2 x + \log_2(x-4)] = \frac{1}{3}\{\log_2[x(x-4)]\}$ Product Property

$= \log_2[x(x-4)]^{1/3}$ Power Property

$= \log_2 \sqrt[3]{x(x-4)}$ Rewrite with a radical.

CHECKPOINT Now try Exercise 81.

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Application

Example 7 Finding a Mathematical Model

The table shows the mean distance x from the sun and the period y (the time it takes a planet to orbit the sun) for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where the Earth's mean distance is defined as 1.0), and the period is given in years. The points in the table are plotted in Figure 3.22. Find an equation that relates y and x.

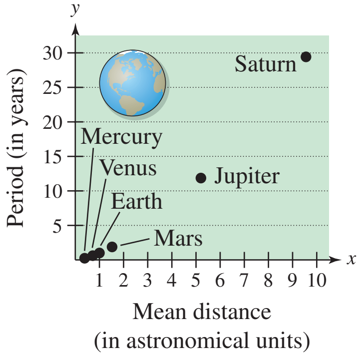

Figure 3.22

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Mean distance, x	0.387	0.723	1.000	1.524	5.203	9.555
Period, y	0.241	0.615	1.000	1.881	11.860	29.420

Solution

From Figure 3.22, it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values in the table. This produces the following results.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
$\ln x = X$	-0.949	-0.324	0.000	0.421	1.649	2.257
$\ln y = Y$	-1.423	-0.486	0.000	0.632	2.473	3.382

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line, as shown in Figure 3.23. To find an equation of the line through these points, you can use one of the following methods.

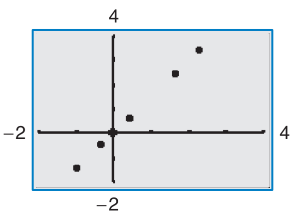

Figure 3.23

Method 1: Algebraic

Choose any two points to determine the slope of the line. Using the two points $(0.421, 0.632)$ and $(0, 0)$, you can determine that the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is

$$Y = \tfrac{3}{2}X$$

where $Y = \ln y$ and $X = \ln x$. You can therefore conclude that

$$\ln y = \frac{3}{2} \ln x.$$

CHECKPOINT Now try Exercise 109.

Method 2: Graphical

Using the *linear regression* feature of a graphing utility, you can find a linear model for the data, as shown in Figure 3.24. You can approximate this model to be $Y = 1.5X$, where $Y = \ln y$ and $X = \ln x$. From the model, you can see that the slope of the line is $\frac{3}{2}$. So, you can conclude that

$$\ln y = \tfrac{3}{2} \ln x.$$

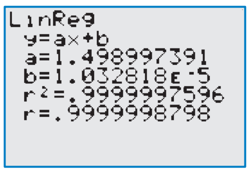

Figure 3.24

In Example 7, try to convert the final equation to $y = f(x)$ form. You will get a function of the form $y = ax^b$, which is called a *power model*.

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3.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. You can evaluate logarithms to any base using the _______ formula.
2. Two properties of logarithms are _______ $= n \log_a u$ and $\ln(uv) =$ _______.

3. Is $\log_3 24 = \dfrac{\ln 3}{\ln 24}$ or $\log_3 24 = \dfrac{\ln 24}{\ln 3}$ correct?
4. Which property of logarithms can you use to condense the expression $\ln x - \ln 2$?

Procedures and Problem Solving

Changing the Base **In Exercises 5–12, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.**

5. $\log_5 x$
6. $\log_3 x$
7. $\log_{1/5} x$
8. $\log_{1/3} x$
9. $\log_a \frac{3}{10}$
10. $\log_a \frac{4}{5}$
11. $\log_{2.6} x$
12. $\log_{7.1} x$

Changing the Base **In Exercises 13–20, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.**

✓ 13. $\log_3 7$
14. $\log_7 4$
15. $\log_{1/2} 4$
16. $\log_{1/8} 64$
17. $\log_6 0.9$
18. $\log_4 0.045$
✓ 19. $\log_{15} 1460$
20. $\log_{20} 175$

Using Properties of Logarithms **In Exercises 21–24, rewrite the expression in terms of ln 4 and ln 5.**

✓ 21. $\ln 20$
22. $\ln 500$
23. $\ln \frac{25}{4}$
24. $\ln \frac{5}{2}$

Using Properties to Evaluate Logarithms **In Exercises 25–28, approximate the logarithm using the properties of logarithms, given the values $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$. Round your result to four decimal places.**

25. $\log_b 25$
26. $\log_b 30$
27. $\log_b \sqrt{3}$
28. $\log_b \frac{16}{25}$

Graphing a Logarithm **In Exercises 29–36, use the change-of-base formula $\log_a x = (\ln x)/(\ln a)$ and a graphing utility to graph the function.**

29. $f(x) = \log_3(x+2)$
30. $f(x) = \log_2(x-1)$
31. $f(x) = \log_{1/2}(x-2)$
32. $f(x) = \log_{1/3}(x+1)$
33. $f(x) = \log_{1/4} x^2$
34. $f(x) = \log_3 \sqrt{x}$
35. $f(x) = \log_{1/2}\left(\dfrac{x}{2}\right)$
36. $\log_5\left(\dfrac{x}{3}\right)$

Simplifying a Logarithm **In Exercises 37–44, use the properties of logarithms to rewrite and simplify the logarithmic expression.**

37. $\log_4 8$
38. $\log_9 243$
39. $\log_2 4^2 \cdot 3^4$
40. $\log_3 9^2 \cdot 2^4$
41. $\ln 5e^6$
42. $\ln 8e^3$
43. $\ln \dfrac{6}{e^2}$
44. $\ln \dfrac{e^5}{7}$

Using Properties of Logarithms **In Exercises 45 and 46, use the properties of logarithms to verify the equation.**

✓ 45. $\log_5 \frac{1}{250} = -3 - \log_5 2$
46. $-\ln 24 = -(3\ln 2 + \ln 3)$

Expanding Logarithmic Expressions **In Exercises 47–64, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

47. $\log_{10} 5x$
48. $\log_{10} 10z$
49. $\log_{10} \dfrac{t}{8}$
50. $\log_{10} \dfrac{7}{z}$
51. $\log_8 x^4$
52. $\log_6 z^{-3}$
53. $\ln \sqrt{z}$
54. $\ln \sqrt[3]{t}$
55. $\ln xyz$
56. $\ln \dfrac{xy}{z}$
57. $\log_6 ab^3c^2$
58. $\log_4 xy^6z^4$
59. $\ln \sqrt[3]{\dfrac{x}{y}}$
60. $\ln \sqrt{\dfrac{x^2}{y^3}}$
61. $\ln \dfrac{x^2-1}{x^3}, \quad x > 1$
62. $\ln \dfrac{x}{\sqrt{x^2+1}}$
✓ 63. $\ln \dfrac{x^4\sqrt{y}}{z^5}$
64. $\log_b \dfrac{\sqrt{x}y^4}{z^4}$

Algebraic-Graphical-Numerical **In Exercises 65–68, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Verify your conclusion algebraically.**

65. $y_1 = \ln[x^3(x+4)], \quad y_2 = 3 \ln x + \ln(x+4)$

66. $y_1 = \ln\left(\dfrac{\sqrt{x}}{x-2}\right), \quad y_2 = \dfrac{1}{2}\ln x - \ln(x-2)$

67. $y_1 = \ln\left(\dfrac{x^4}{x-2}\right), \quad y_2 = 4 \ln x - \ln(x-2)$

68. $y_1 = \ln 4x^3, \quad y_2 = \ln 4 + 3 \ln x$

Condensing Logarithmic Expressions **In Exercises 69–84, condense the expression to the logarithm of a single quantity.**

69. $\ln x + \ln 4$

70. $\ln y + \ln z$

71. $\log_4 z - \log_4 y$

72. $\log_5 8 - \log_5 t$

73. $2 \log_2(x+3)$

74. $\frac{5}{2} \log_7(z-4)$

75. $\frac{1}{2}\ln(x^2+4)$

76. $2 \ln x + \ln(x+1)$

77. $\ln x - 3 \ln(x+1)$

78. $\ln x - 2 \ln(x+2)$

79. $\ln(x-2) - \ln(x+2)$

80. $3 \ln x + 2 \ln y - 4 \ln z$

81. $\ln x - 2[\ln(x+2) + \ln(x-2)]$

82. $4[\ln z + \ln(z+5)] - 2 \ln(z-5)$

83. $\frac{1}{3}[2 \ln(x+3) + \ln x - \ln(x^2-1)]$

84. $2[\ln x - \ln(x+1) - \ln(x-1)]$

Algebraic-Graphical-Numerical **In Exercises 85–88, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Verify your conclusion algebraically.**

85. $y_1 = 2[\ln 8 - \ln(x^2+1)], \quad y_2 = \ln\left[\dfrac{64}{(x^2+1)^2}\right]$

86. $y_1 = 2[\ln 6 + \ln(x^2+1)], \quad y_2 = \ln[36(x^2+1)^2]$

87. $y_1 = \ln x + \frac{1}{2}\ln(x+1), \quad y_2 = \ln\left(x\sqrt{x+1}\right)$

88. $y_1 = \dfrac{1}{2}\ln x - \ln(x+2), \quad y_2 = \ln\left(\dfrac{\sqrt{x}}{x+2}\right)$

Algebraic-Graphical-Numerical **In Exercises 89–92, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) Are the expressions equivalent? Explain. Verify your conclusion algebraically.**

89. $y_1 = \ln x^2, \quad y_2 = 2 \ln x$

90. $y_1 = 2(\ln 2 + \ln x), \quad y_2 = \ln 4x^2$

91. $y_1 = \ln(x-2) + \ln(x+2), \quad y_2 = \ln(x^2-4)$

92. $y_1 = \frac{1}{4}\ln[x^4(x^2+1)], \quad y_2 = \ln x + \frac{1}{4}\ln(x^2+1)$

Using Properties to Evaluate Logarithms **In Exercises 93–106, find the exact value of the logarithm without using a calculator. If this is not possible, state the reason.**

93. $\log_3 9$

94. $\log_6 \sqrt[3]{6}$

95. $\log_4 16^{3.4}$

96. $\log_5\left(\frac{1}{125}\right)$

97. $\log_2(-4)$

98. $\log_4(-16)$

99. $\log_5 75 - \log_5 3$

100. $\log_4 2 + \log_4 32$

101. $\ln e^3 - \ln e^7$

102. $\ln e^6 - 2 \ln e^5$

103. $2 \ln e^4$

104. $\ln e^{4.5}$

105. $\ln \dfrac{1}{\sqrt{e}}$

106. $\ln \sqrt[5]{e^3}$

107. ***Why you should learn it*** *(p. 203)* The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Use the properties of logarithms to write the formula in a simpler form.

(b) Use a graphing utility to complete the table. Verify your answers algebraically.

I	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
β						

108. **Psychology** Students participating in a psychology experiment attended several lectures and were given an exam. Every month for the next year, the students were retested to see how much of the material they remembered. The average scores for the group are given by the human memory model

$$f(t) = 90 - 15 \log_{10}(t+1), \quad 0 \le t \le 12$$

where t is the time (in months).

(a) Use a graphing utility to graph the function over the specified domain.

(b) What was the average score on the original exam $(t = 0)$?

(c) What was the average score after 6 months?

(d) What was the average score after 12 months?

(e) When did the average score decrease to 75?

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✓ **109. MODELING DATA**

A beaker of liquid at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the liquid is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T), where t is the time (in minutes) and T is the temperature (in degrees Celsius).

$(0, 78.0°)$, $(5, 66.0°)$, $(10, 57.5°)$, $(15, 51.2°)$, $(20, 46.3°)$, $(25, 42.5°)$, $(30, 39.6°)$

(a) The graph of the temperature of the room should be an asymptote of the graph of the model for the data. Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T - 21)$.

(b) An exponential model for the data $(t, T - 21)$ is given by

$$T - 21 = 54.4(0.964)^t.$$

Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Take the natural logarithms of the revised temperatures. Use the graphing utility to plot the points $(t, \ln(T - 21))$ and observe that the points appear linear. Use the *regression* feature of the graphing utility to fit a line to the data. The resulting line has the form

$$\ln(T - 21) = at + b.$$

Use the properties of logarithms to solve for T. Verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y-coordinates of the revised data points to generate the points

$$\left(t, \frac{1}{T - 21}\right).$$

Use the graphing utility to plot these points and observe that they appear linear. Use the *regression* feature of the graphing utility to fit a line to the data. The resulting line has the form

$$\frac{1}{T - 21} = at + b.$$

Solve for T, and use the graphing utility to graph the rational function and the original data points.

110. Writing Write a short paragraph explaining why the transformations of the data in Exercise 109 were necessary to obtain the models. Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Conclusions

True or False? In Exercises 111–116, determine whether the statement is true or false given that $f(x) = \ln x$, where $x > 0$. Justify your answer.

111. $f(ax) = f(a) + f(x), \quad a > 0$

112. $f(x - a) = f(x) - f(a), \quad x > a$

113. $\sqrt{f(x)} = \frac{1}{2}f(x)$

114. $[f(x)]^n = nf(x)$

115. If $f(x) < 0$, then $0 < x < e$.

116. If $f(x) > 0$, then $x > e$.

117. Error Analysis Describe the error.

$$\ln\left(\frac{x^2}{\sqrt{x^2 + 4}}\right) = \frac{\ln x^2}{\ln\sqrt{x^2 + 4}}$$

118. Think About It Consider the functions below.

$$f(x) = \ln\frac{x}{2}, \quad g(x) = \frac{\ln x}{\ln 2}, \quad h(x) = \ln x - \ln 2$$

Which two functions have identical graphs? Verify your answer by using a graphing utility to graph all three functions in the same viewing window.

119. Exploration For how many integers between 1 and 20 can the natural logarithms be approximated given that $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms. (Do not use a calculator.)

120. CAPSTONE Show that each expression is equivalent to ln 8. Then write three more expressions that are equivalent to ln 8.

(a) $3 \ln 2$ (b) $-\ln \frac{1}{8}$ (c) $-3 \ln \frac{1}{2}$

(d) $\ln 16 - \ln 2$ (e) $(\log_{10} 2 + \log_{10} 4) \div \log_{10} e$

121. Think About It Does $y_1 = \ln[x(x - 2)]$ have the same domain as $y_2 = \ln x + \ln(x - 2)$? Explain.

122. Proof Prove that $\dfrac{\log_a x}{\log_{a/b} x} = 1 + \log_a \dfrac{1}{b}$.

Cumulative Mixed Review

Using Rules of Exponents In Exercises 123–126, simplify the expression.

123. $\dfrac{24xy^{-2}}{16x^{-3}y}$

124. $\left(\dfrac{2x^3}{3y}\right)^{-3}$

125. $(18x^3y^4)^{-3}(18x^3y^4)^4$

126. $xy(x^{-1} + y^{-1})^{-1}$

Solving Polynomial Equations In Exercises 127–130, find all solutions of the equation.

127. $x^2 - 6x + 2 = 0$

128. $2x^3 + 20x^2 + 50x = 0$

129. $x^4 - 19x^2 + 48 = 0$

130. $9x^4 - 37x^2 + 4 = 0$

3.4 Solving Exponential and Logarithmic Equations

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which

$\log_a x$ and $\log_a y$

are defined.

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$a^{\log_a x} = x$

$\log_a a^x = x$

What you should learn

- Solve simple exponential and logarithmic equations.
- Solve more complicated exponential equations.
- Solve more complicated logarithmic equations.
- Use exponential and logarithmic equations to model and solve real-life problems.

Why you should learn it

Exponential and logarithmic equations can be used to model and solve real-life problems. For instance, Exercise 148 on page 219 shows how to use an exponential function to model the average heights of men and women.

Example 1 Solving Simple Exponential and Logarithmic Equations

	Original Equation	Rewritten Equation	Solution	Property
a.	$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b.	$\log_4 x - \log_4 8 = 0$	$\log_4 x = \log_4 8$	$x = 8$	One-to-One
c.	$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
d.	$\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
e.	$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
f.	$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
g.	$\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
h.	$\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

CHECKPOINT Now try Exercise 27.

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

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Solving Exponential Equations

Example 2 Solving Exponential Equations

Solve each equation.

a. $e^x = 72$

b. $3(2^x) = 42$

Algebraic Solution

a.

$$\begin{aligned} e^x &= 72 && \text{Write original equation.} \\ \ln e^x &= \ln 72 && \text{Take natural log of each side.} \\ x &= \ln 72 && \text{Inverse Property} \\ x &\approx 4.28 && \text{Use a calculator.} \end{aligned}$$

The solution is $x = \ln 72 \approx 4.28$. Check this in the original equation.

b.

$$\begin{aligned} 3(2^x) &= 42 && \text{Write original equation.} \\ 2^x &= 14 && \text{Divide each side by 3.} \\ \log_2 2^x &= \log_2 14 && \text{Take log (base 2) of each side.} \\ x &= \log_2 14 && \text{Inverse Property} \\ x &= \frac{\ln 14}{\ln 2} && \text{Change-of-base formula} \\ x &\approx 3.81 && \text{Use a calculator.} \end{aligned}$$

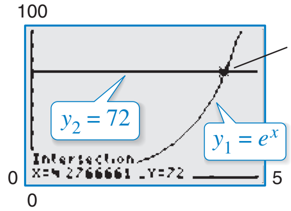
The solution is $x = \log_2 14 \approx 3.81$. Check this in the original equation.

CHECKPOINT Now try Exercise 33.

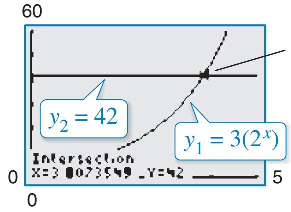
Graphical Solution

To solve an equation using a graphing utility, you can graph the left- and right-hand sides of the equation and use the *intersect* feature.

a.



b.



Example 3 Solving an Exponential Equation

Solve $4e^{2x} - 3 = 2$.

Algebraic Solution

$$\begin{aligned} 4e^{2x} - 3 &= 2 && \text{Write original equation.} \\ 4e^{2x} &= 5 && \text{Add 3 to each side.} \\ e^{2x} &= \tfrac{5}{4} && \text{Divide each side by 4.} \\ \ln e^{2x} &= \ln \tfrac{5}{4} && \text{Take natural log of each side.} \\ 2x &= \ln \tfrac{5}{4} && \text{Inverse Property} \\ x &= \tfrac{1}{2} \ln \tfrac{5}{4} && \text{Divide each side by 2.} \\ x &\approx 0.11 && \text{Use a calculator.} \end{aligned}$$

The solution is

$$x = \tfrac{1}{2} \ln \tfrac{5}{4} \approx 0.11.$$

Check this in the original equation.

CHECKPOINT Now try Exercise 61.

Graphical Solution

Rather than using the procedure in Example 2, another way to solve the equation graphically is first to rewrite the equation as

$$4e^{2x} - 5 = 0$$

and then use a graphing utility to graph

$$y = 4e^{2x} - 5.$$

Use the *zero* or *root* feature of the graphing utility to approximate the value of x for which $y = 0$, as shown in Figure 3.25.

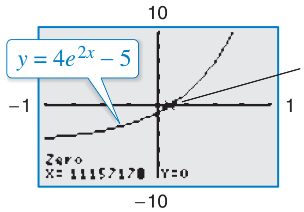


Figure 3.25

Example 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$.

Solution

$2(3^{2t-5}) - 4 = 11$	Write original equation.
$2(3^{2t-5}) = 15$	Add 4 to each side.
$3^{2t-5} = \frac{15}{2}$	Divide each side by 2.
$\log_3 3^{2t-5} = \log_3 \frac{15}{2}$	Take log (base 3) of each side.
$2t - 5 = \log_3 \frac{15}{2}$	Inverse Property
$2t = 5 + \log_3 7.5$	Add 5 to each side.
$t = \frac{5}{2} + \frac{1}{2}\log_3 7.5$	Divide each side by 2.
$t \approx 3.42$	Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.42$. Check this in the original equation.

CHECKPOINT Now try Exercise 65.

Study Tip

Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in the previous three examples. However, the algebra is a bit more complicated.

Example 5 Solving an Exponential Equation in Quadratic Form

Solve $e^{2x} - 3e^x + 2 = 0$.

Algebraic Solution

$e^{2x} - 3e^x + 2 = 0$	Write original equation.
$(e^x)^2 - 3e^x + 2 = 0$	Write in quadratic form.
$(e^x - 2)(e^x - 1) = 0$	Factor.
$e^x - 2 = 0$	Set 1st factor equal to 0.
$e^x = 2$	Add 2 to each side.
$x = \ln 2$	Solution
$e^x - 1 = 0$	Set 2nd factor equal to 0.
$e^x = 1$	Add 1 to each side.
$x = \ln 1$	Inverse Property
$x = 0$	Solution

The solutions are

$$x = \ln 2 \approx 0.69 \text{ and } x = 0.$$

Check these in the original equation.

CHECKPOINT Now try Exercise 71.

Graphical Solution

Use a graphing utility to graph $y = e^{2x} - 3e^x + 2$ and then find the zeros.

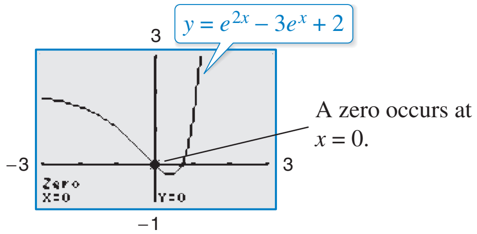


Figure 3.26

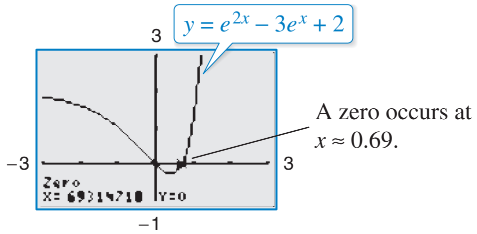


Figure 3.27

From Figures 3.26 and 3.27, you can conclude that the solutions are $x = 0$ and $x \approx 0.69$.

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Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$\ln x = 3$ Logarithmic form

$e^{\ln x} = e^3$ Exponentiate each side.

$x = e^3$ Exponential form

This procedure is called *exponentiating* each side of an equation. It is applied after the logarithmic expression has been isolated.

Example 6 Solving Logarithmic Equations

Solve each logarithmic equation.

a. $\ln 3x = 2$

b. $\log_3(5x - 1) = \log_3(x + 7)$

Solution

a. $\ln 3x = 2$ Write original equation.

$e^{\ln 3x} = e^2$ Exponentiate each side.

$3x = e^2$ Inverse Property

$x = \frac{1}{3}e^2$ Multiply each side by $\frac{1}{3}$.

$x \approx 2.46$ Use a calculator.

The solution is $x = \frac{1}{3}e^2 \approx 2.46$. Check this in the original equation.

b. $\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.

$5x - 1 = x + 7$ One-to-One Property

$x = 2$ Solve for x.

The solution is $x = 2$. Check this in the original equation.

CHECKPOINT Now try Exercise 93.

Example 7 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$.

Algebraic Solution

$5 + 2 \ln x = 4$ Write original equation.

$2 \ln x = -1$ Subtract 5 from each side.

$\ln x = -\frac{1}{2}$ Divide each side by 2.

$e^{\ln x} = e^{-1/2}$ Exponentiate each side.

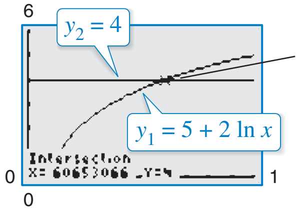
$x = e^{-1/2}$ Inverse Property

$x \approx 0.61$ Use a calculator.

The solution is $x = e^{-1/2} \approx 0.61$. Check this in the original equation.

CHECKPOINT Now try Exercise 97.

Graphical Solution



The intersection point is about (0.61, 4). So, the solution is $x \approx 0.61$.

Example 8 Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$2 \log_5 3x = 4$	Write original equation.
$\log_5 3x = 2$	Divide each side by 2.
$5^{\log_5 3x} = 5^2$	Exponentiate each side (base 5).
$3x = 25$	Inverse Property
$x = \frac{25}{3}$	Divide each side by 3.

The solution is $x = \frac{25}{3}$. Check this in the original equation. Or, perform a graphical check by graphing

$$y_1 = 2 \log_5 3x = 2\left(\frac{\log_{10} 3x}{\log_{10} 5}\right) \quad \text{and} \quad y_2 = 4$$

in the same viewing window. The two graphs should intersect at

$$x = \tfrac{25}{3} \approx 8.33$$

and

$$y = 4$$

as shown in Figure 3.28.

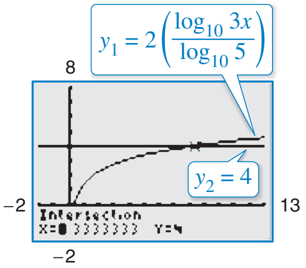


Figure 3.28

CHECKPOINT Now try Exercise 99.

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations, as shown in the next example.

Example 9 Checking for Extraneous Solutions

Solve $\ln(x - 2) + \ln(2x - 3) = 2 \ln x$.

Algebraic Solution

$\ln(x - 2) + \ln(2x - 3) = 2 \ln x$	Write original equation.
$\ln[(x - 2)(2x - 3)] = \ln x^2$	Use properties of logarithms.
$\ln(2x^2 - 7x + 6) = \ln x^2$	Multiply binomials.
$2x^2 - 7x + 6 = x^2$	One-to-One Property
$x^2 - 7x + 6 = 0$	Write in general form.
$(x - 6)(x - 1) = 0$	Factor.
$x - 6 = 0 \Rightarrow x = 6$	Set 1st factor equal to 0.
$x - 1 = 0 \Rightarrow x = 1$	Set 2nd factor equal to 0.

Finally, by checking these two "solutions" in the original equation, you can conclude that $x = 1$ is not valid. This is because when $x = 1$,

$$\ln(x - 2) + \ln(2x - 3) = \ln(-1) + \ln(-1)$$

which is invalid because -1 is not in the domain of the natural logarithmic function. So, the only solution is $x = 6$.

CHECKPOINT Now try Exercise 109.

Graphical Solution

First rewrite the original equation as

$$\ln(x - 2) + \ln(2x - 3) - 2 \ln x = 0.$$

Then use a graphing utility to graph the equation

$$y = \ln(x - 2) + \ln(2x - 3) - 2 \ln x$$

and find the zeros (see Figure 3.29).

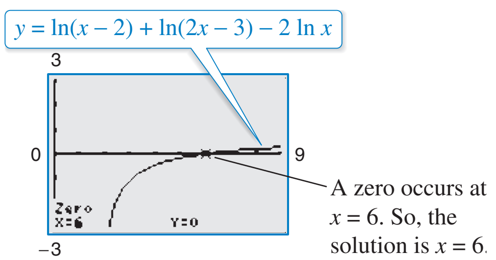


Figure 3.29

Example 10 The Change-of-Base Formula

Prove the change-of-base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Solution

Begin by letting

$$y = \log_a x$$

and writing the equivalent exponential form

$$a^y = x.$$

Now, taking the logarithms *with base b* of each side produces the following.

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x \qquad \text{Power Property}$$

$$y = \frac{\log_b x}{\log_b a} \qquad \text{Divide each side by } \log_b a.$$

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{Replace } y \text{ with } \log_a x.$$

CHECKPOINT Now try Exercise 113.

Equations that involve combinations of algebraic functions, exponential functions, and/or logarithmic functions can be very difficult to solve by algebraic procedures. Here again, you can take advantage of a graphing utility.

Example 11 Approximating the Solution of an Equation

Approximate (to three decimal places) the solution of $\ln x = x^2 - 2$.

Solution

First, rewrite the equation as

$$\ln x - x^2 + 2 = 0.$$

Then use a graphing utility to graph

$$y = -x^2 + 2 + \ln x$$

as shown in Figure 3.30. From this graph, you can see that the equation has two solutions. Next, using the *zero* or *root* feature, you can approximate the two solutions to be $x \approx 0.138$ and $x \approx 1.564$.

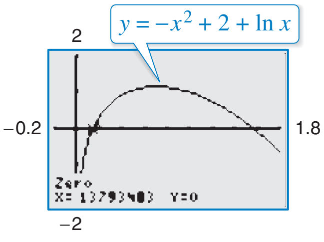


Figure 3.30

Check

$$\ln x = x^2 - 2 \qquad \text{Write original equation.}$$

$$\ln(0.138) \stackrel{?}{\approx} (0.138)^2 - 2 \qquad \text{Substitute 0.138 for } x.$$

$$-1.9805 \approx -1.9810 \qquad \text{Solution checks. } \checkmark$$

$$\ln(1.564) \stackrel{?}{\approx} (1.564)^2 - 2 \qquad \text{Substitute 1.564 for } x.$$

$$0.4472 \approx 0.4461 \qquad \text{Solution checks. } \checkmark$$

So, the two solutions $x \approx 0.138$ and $x \approx 1.564$ seem reasonable.

CHECKPOINT Now try Exercise 119.

Applications

Example 12 Doubling an Investment

You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution

Using the formula for continuous compounding, you can find that the balance in the account is

$$A = Pe^{rt} = 500e^{0.0675t}.$$

To find the time required for the balance to double, let $A = 1000$, and solve the resulting equation for t.

$500e^{0.0675t} = 1000$	Substitute 1000 for A.
$e^{0.0675t} = 2$	Divide each side by 500.
$\ln e^{0.0675t} = \ln 2$	Take natural log of each side.
$0.0675t = \ln 2$	Inverse Property
$t = \dfrac{\ln 2}{0.0675}$	Divide each side by 0.0675.
$t \approx 10.27$	Use a calculator.

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically in Figure 3.31.

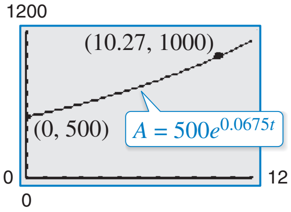


Figure 3.31

CHECKPOINT Now try Exercise 143.

Example 13 Average Salary for Public School Teachers

From 1985 through 2007, the average salary y (in thousands of dollars) for public school teachers for the year t can be modeled by the equation

$$y = -2.983 + 15.206 \ln t, \quad 5 \le t \le 27$$

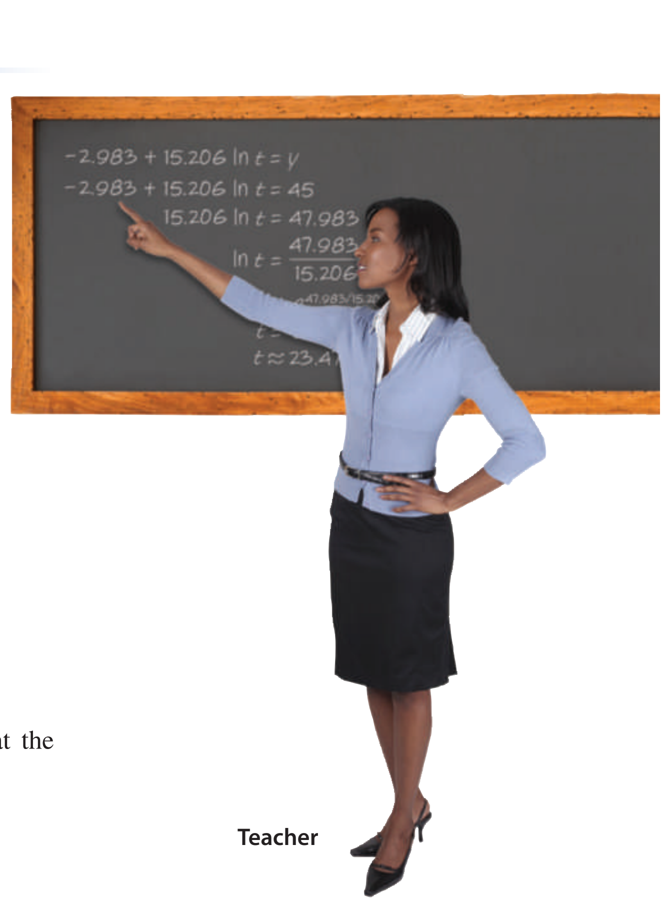
where $t = 5$ represents 1985. During which year did the average salary for public school teachers reach \$45,000? (Source: National Education Association)

Solution

$-2.983 + 15.206 \ln t = y$	Write original equation.
$-2.983 + 15.206 \ln t = 45$	Substitute 45 for y.
$15.206 \ln t = 47.983$	Add 2.983 to each side.
$\ln t = \dfrac{47.983}{15.206}$	Divide each side by 15.206.
$e^{\ln t} = e^{47.983/15.206}$	Exponentiate each side.
$t = e^{47.983/15.206}$	Inverse Property
$t \approx 23.47$	Use a calculator.

The solution is $t \approx 23.47$ years. Because $t = 5$ represents 1985, it follows that the average salary for public school teachers reached \$45,000 in 2003.

CHECKPOINT Now try Exercise 149.



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3.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercise 1 and 2, fill in the blank.

1. To solve exponential and logarithmic equations, you can use the following One-to-One and Inverse Properties.

(a) $a^x = a^y$ if and only if _______ . (b) $\log_a x = \log_a y$ if and only if _______ .

(c) $a^{\log_a x} =$ _______ (d) $\log_a a^x =$ _______

2. An _______ solution does not satisfy the original equation.

3. What is the value of $\ln e^7$?

4. Can you solve $5^x = 125$ using a One-to-One Property?

5. What is the first step in solving the equation $3 + \ln x = 10$?

6. Do you solve $\log_4 x = 2$ by using a One-to-One Property or an Inverse Property?

Procedures and Problem Solving

Checking Solutions **In Exercises 7–14, determine whether each x-value is a solution of the equation.**

7. $4^{2x-7} = 64$
(a) $x = 5$
(b) $x = 2$

8. $2^{3x+1} = 32$
(a) $x = -1$
(b) $x = 2$

9. $3e^{x+2} = 75$
(a) $x = -2 + e^{25}$
(b) $x = -2 + \ln 25$
(c) $x \approx 1.2189$

10. $4e^{x-1} = 60$
(a) $x = 1 + \ln 15$
(b) $x \approx 3.7081$
(c) $x = \ln 16$

11. $\log_4(3x) = 3$
(a) $x \approx 21.3560$
(b) $x = -4$
(c) $x = \frac{64}{3}$

12. $\log_6\left(\frac{5}{3}x\right) = 2$
(a) $x \approx 20.2882$
(b) $x = \frac{108}{5}$
(c) $x = 7.2$

13. $\ln(x - 1) = 3.8$
(a) $x = 1 + e^{3.8}$
(b) $x \approx 45.7012$
(c) $x = 1 + \ln 3.8$

14. $\ln(2 + x) = 2.5$
(a) $x = e^{2.5} - 2$
(b) $x \approx \frac{4073}{400}$
(c) $x = \frac{1}{2}$

Solving Equations Graphically **In Exercises 15–22, use a graphing utility to graph f and g in the same viewing window. Approximate the point of intersection of the graphs of f and g. Then solve the equation $f(x) = g(x)$ algebraically.**

15. $f(x) = 2^x$
$g(x) = 8$

16. $f(x) = 27^x$
$g(x) = 9$

17. $f(x) = 5^{x-2} - 15$
$g(x) = 10$

18. $f(x) = 2^{-x+1} - 3$
$g(x) = 13$

19. $f(x) = 4\log_3 x$
$g(x) = 20$

20. $f(x) = 3\log_5 x$
$g(x) = 6$

21. $f(x) = \ln e^{x+1}$
$g(x) = 2x + 5$

22. $f(x) = \ln e^{x-2}$
$g(x) = 3x + 2$

Solving an Exponential Equation **In Exercises 23–36, solve the exponential equation.**

23. $4^x = 16$ **24.** $3^x = 243$

25. $5^x = \frac{1}{625}$ **26.** $7^x = \frac{1}{49}$

✓ **27.** $\left(\frac{1}{8}\right)^x = 64$ **28.** $\left(\frac{1}{2}\right)^x = 32$

29. $\left(\frac{2}{3}\right)^x = \frac{81}{16}$ **30.** $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

31. $e^x = 14$ **32.** $e^x = 66$

✓ **33.** $6(10^x) = 216$ **34.** $5(8^x) = 325$

35. $2^{x+3} = 256$ **36.** $3^{x-1} = \frac{1}{81}$

Solving a Logarithmic Equation **In Exercises 37–46, solve the logarithmic equation.**

37. $\ln x - \ln 5 = 0$ **38.** $\ln x - \ln 2 = 0$

39. $\ln x = -9$ **40.** $\ln x = -14$

41. $\log_x 625 = 4$ **42.** $\log_x 25 = 2$

43. $\log_{10} x = -1$ **44.** $\log_{10} x = -\frac{1}{2}$

45. $\ln(2x - 1) = 5$ **46.** $\ln(3x + 5) = 8$

Using Inverse Properties **In Exercises 47–54, simplify the expression.**

47. $\ln e^{x^2}$ **48.** $\ln e^{2x-1}$

49. $e^{\ln x^2}$ **50.** $e^{\ln(x^2+2)}$

51. $-1 + \ln e^{2x}$

52. $-4 + e^{\ln x^4}$

53. $5 + e^{\ln(x^2+1)}$

54. $3 - \ln(e^{x^2+2})$

Solving an Exponential Equation In Exercises 55–80, solve the exponential equation algebraically. Round your result to three decimal places. Use a graphing utility to verify your answer.

55. $8^{3x} = 360$
56. $6^{5x} = 3000$
57. $5^{-t/2} = 0.20$
58. $4^{-3t} = 0.10$
59. $250e^{0.02x} = 10{,}000$
60. $100e^{0.005x} = 125{,}000$
61. $500e^{-x} = 300$
62. $1000e^{-4x} = 75$
63. $7 - 2e^x = 5$
64. $-14 + 3e^x = 11$
65. $5(2^{3-x}) - 13 = 100$
66. $6(8^{-2-x}) + 15 = 2601$
67. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$
68. $\left(16 + \dfrac{0.878}{26}\right)^{3t} = 30$
69. $5000\left[\dfrac{(1 + 0.005)^x}{0.005}\right] = 250{,}000$
70. $250\left[\dfrac{(1 + 0.01)^x}{0.01}\right] = 150{,}000$
71. $e^{2x} - 4e^x - 5 = 0$
72. $e^{2x} - 5e^x + 6 = 0$
73. $e^x = e^{x^2-2}$
74. $e^{2x} = e^{x^2-8}$
75. $e^{x^2-3x} = e^{x-2}$
76. $e^{-x^2} = e^{x^2-2x}$
77. $\dfrac{400}{1 + e^{-x}} = 350$
78. $\dfrac{525}{1 + e^{-x}} = 275$
79. $\dfrac{40}{1 - 5e^{-0.01x}} = 200$
80. $\dfrac{50}{1 - 2e^{-0.001x}} = 1000$

Algebraic-Graphical-Numerical In Exercises 81 and 82, (a) complete the table to find an interval containing the solution of the equation, (b) use a graphing utility to graph both sides of the equation to estimate the solution, and (c) solve the equation algebraically. Round your results to three decimal places.

81. $e^{3x} = 12$

x	0.6	0.7	0.8	0.9	1.0
e^{3x}					

82. $20(100 - e^{x/2}) = 500$

x	5	6	7	8	9
$20(100 - e^{x/2})$					

Solving an Exponential Equation Graphically In Exercises 83–86, use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to approximate the solution of the exponential equation accurate to three decimal places.

83. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$
84. $\left(4 - \dfrac{2.471}{40}\right)^{9t} = 21$
85. $\dfrac{3000}{2 + e^{2x}} = 2$
86. $\dfrac{119}{e^{6x} - 14} = 7$

Finding the Zero of a Function In Exercises 87–90, use a graphing utility to graph the function and approximate its zero accurate to three decimal places.

87. $g(x) = 6e^{1-x} - 25$
88. $f(x) = 3e^{3x/2} - 962$
89. $g(t) = e^{0.09t} - 3$
90. $h(t) = e^{0.125t} - 8$

Solving a Logarithmic Equation In Exercises 91–112, solve the logarithmic equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.

91. $\ln x = -3$
92. $\ln x = -4$
93. $\ln 4x = 2.1$
94. $\ln 2x = 1.5$
95. $\log_5(3x + 2) = \log_5(6 - x)$
96. $\log_9(4 + x) = \log_9(2x - 1)$
97. $-2 + 2\ln 3x = 17$
98. $3 + 2\ln x = 10$
99. $7\log_4(0.6x) = 12$
100. $4\log_{10}(x - 6) = 11$
101. $\log_{10}(z - 3) = 2$
102. $\log_{10} x^2 = 6$
103. $\ln\sqrt{x + 2} = 1$
104. $\ln\sqrt{x - 8} = 5$
105. $\ln(x + 1)^2 = 2$
106. $\ln(x^2 + 1) = 8$
107. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$
108. $\log_3 x + \log_3(x - 8) = 2$
109. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$
110. $\ln(x + 1) - \ln(x - 2) = \ln x$
111. $\log_{10} 8x - \log_{10}(1 + \sqrt{x}) = 2$
112. $\log_{10} 4x - \log_{10}(12 + \sqrt{x}) = 2$

The Change-of-Base Formula In Exercises 113 and 114, use the method of Example 10 to prove the change-of-base formula for the indicated base.

113. $\log_a x = \dfrac{\log_e x}{\log_e a}$
114. $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

Algebraic-Graphical-Numerical In Exercises 115–118, (a) complete the table to find an interval containing the solution of the equation, (b) use a graphing utility to graph both sides of the equation to estimate the solution, and (c) solve the equation algebraically. Round your results to three decimal places.

115. $\ln 2x = 2.4$

x	2	3	4	5	6
$\ln 2x$					

116. $3\ln 5x = 10$

x	4	5	6	7	8
$3\ln 5x$					

117. $6 \log_3(0.5x) = 11$

x	12	13	14	15	16
$6 \log_3(0.5x)$					

118. $5 \log_{10}(x - 2) = 11$

x	150	155	160	165	170
$5 \log_{10}(x - 2)$					

Approximating the Solution of an Equation **In Exercises 119–124, use the *zero* or *root* feature of a graphing utility to approximate the solution of the logarithmic equation.**

✓ **119.** $\log_{10} x = x^3 - 3$ **120.** $\log_{10} x^2 = 4$

121. $\ln x + \ln(x - 2) = 1$ **122.** $\ln x + \ln(x + 1) = 2$

123. $\ln(x - 3) + \ln(x + 3) = 1$

124. $\ln x + \ln(x^2 + 4) = 10$

Finding the Point of Intersection **In Exercises 125–130, use a graphing utility to approximate the point of intersection of the graphs. Round your result to three decimal places.**

125. $y_1 = 7$, $y_2 = 2^{x-1} - 5$ **126.** $y_1 = 4$, $y_2 = 3^{x+1} - 2$

127. $y_1 = 80$, $y_2 = 4e^{-0.2x}$ **128.** $y_1 = 500$, $y_2 = 1500e^{-x/2}$

129. $y_1 = 3.25$, $y_2 = \frac{1}{2}\ln(x + 2)$ **130.** $y_1 = 1.05$, $y_2 = \ln\sqrt{x - 2}$

Solving Exponential and Logarithmic Equations **In Exercises 131–138, solve the equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.**

131. $2x^2e^{2x} + 2xe^{2x} = 0$ **132.** $-x^2e^{-x} + 2xe^{-x} = 0$

133. $-xe^{-x} + e^{-x} = 0$ **134.** $e^{-2x} - 2xe^{-2x} = 0$

135. $2x \ln x + x = 0$ **136.** $\dfrac{1 - \ln x}{x^2} = 0$

137. $\dfrac{1 + \ln x}{2} = 0$ **138.** $2x \ln\left(\dfrac{1}{x}\right) - x = 0$

Solving a Model for x **In Exercises 139–142, the equation represents the given type of model, which you will use in Section 3.5. Solve the equation for x.**

Model type	*Equation*
139. Exponential growth	$y = ae^{bx}$
140. Exponential decay	$y = ae^{-bx}$
141. Gaussian	$y = ae^{-(x-b)^2/c}$
142. Logarithmic	$y = a + b \ln x$

Doubling and Tripling an Investment **In Exercises 143–146, find the time required for a \$1000 investment to (a) double at interest rate r, compounded continuously, and (b) triple at interest rate r, compounded continuously. Round your results to two decimal places.**

✓ **143.** $r = 7.5\%$ **144.** $r = 6\%$

145. $r = 2.5\%$ **146.** $r = 3.75\%$

147. Economics The demand x for a handheld electronic organizer is given by

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right)$$

where p is the price in dollars. Find the demands x for prices of (a) $p = \$300$ and (b) $p = \$250$.

148. ***Why you should learn it*** *(p. 210)* The percent m of American males between the ages of 18 and 24 who are no more than x inches tall is modeled by

$$m(x) = \frac{100}{1 + e^{-0.6114(x-69.71)}}$$

and the percent f of American females between the ages of 18 and 24 who are no more than x inches tall is modeled by

$$f(x) = \frac{100}{1 + e^{-0.66607(x-64.51)}}.$$

(Source: U.S. National Center for Health Statistics)

(a) Use a graphing utility to graph the two functions in the same viewing window.

(b) Use the graphs in part (a) to determine the horizontal asymptotes of the functions. Interpret their meanings in the context of the problem.

(c) What is the average height for each sex?

✓ **149. Finance** The numbers y of commercial banks in the United States from 1999 through 2009 can be modeled by

$$y = 13{,}107 - 2077.6 \ln t, \quad 9 \le t \le 19$$

where t represents the year, with $t = 9$ corresponding to 1999. In what year were there about 7100 commercial banks? (Source: Federal Deposit Insurance Corp.)

150. Forestry The yield V (in millions of cubic feet per acre) for a forest at age t years is given by $V = 6.7e^{-48.1/t}$.

(a) Use a graphing utility to graph the function.

(b) Determine the horizontal asymptote of the function. Interpret its meaning in the context of the problem.

(c) Find the time necessary to obtain a yield of 1.3 million cubic feet.

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151. Science An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C. The temperature T of the object was measured after each hour h and recorded in the table. A model for the data is given by $T = 20[1 + 7(2^{-h})]$.

Hour, h	Temperature
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

(a) Use a graphing utility to plot the data and graph the model in the same viewing window.

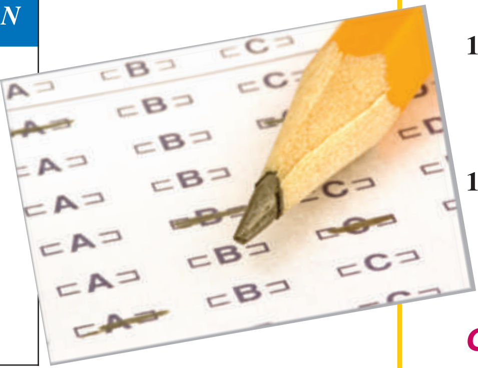
(b) Identify the horizontal asymptote of the graph. Interpret its meaning in the context of the problem.

(c) Approximate the time when the temperature of the object is 100°C.

152. MODELING DATA

The table shows the numbers N of college-bound seniors intending to major in computer or information sciences who took the SAT exam from 2001 through 2009. The data can be modeled by the logarithmic function $N = 77{,}010 - 21{,}554.3 \ln t$, where t represents the year, with $t = 1$ corresponding to 2001. (Source: The College Board)

Year	Number, N
2001	73,466
2002	68,051
2003	53,449
2004	45,879
2005	42,890
2006	37,943
2007	33,965
2008	30,495
2009	31,022



(a) According to the model, in what year would 25,325 seniors intending to major in computer or information sciences take the SAT exam?

(b) Use a graphing utility to graph the model with the data, and use the graph to verify your answer in part (a).

(c) Do you think this is a good model for predicting future values? Explain.

Conclusions

True or False? **In Exercises 153 and 154, determine whether the statement is true or false. Justify your answer.**

153. An exponential equation must have at least one solution.

154. A logarithmic equation can have at most one extraneous solution.

155. Error Analysis Describe the error.

$$\begin{aligned} 2e^x &= 10 \\ \ln(2e^x) &= \ln 10 \\ 2x &= \ln 10 \\ x &= \tfrac{1}{2}\ln 10 \end{aligned}$$

156. CAPSTONE Write two or three sentences stating the general guidelines that you follow when you solve (a) exponential equations and (b) logarithmic equations.

157. Think About It Would you use a One-to-One Property or an Inverse Property to solve $5^x = 34$? Explain.

158. Exploration Let $f(x) = \log_a x$ and $g(x) = a^x$, where $a > 1$.

(a) Let $a = 1.2$ and use a graphing utility to graph the two functions in the same viewing window. What do you observe? Approximate any points of intersection of the two graphs.

(b) Determine the value(s) of a for which the two graphs have one point of intersection.

(c) Determine the value(s) of a for which the two graphs have two points of intersection.

159. Think About It Is the time required for a continuously compounded investment to quadruple twice as long as the time required for it to double? Give a reason for your answer and verify your answer algebraically.

160. Writing Write a paragraph explaining whether or not the time required for a continuously compounded investment to double is dependent on the size of the investment.

Cumulative Mixed Review

Sketching Graphs **In Exercises 161–166, sketch the graph of the function.**

161. $f(x) = 3x^3 - 4$

162. $f(x) = -(x + 1)^3 + 2$

163. $f(x) = |x| + 9$

164. $f(x) = |x + 2| - 8$

165. $f(x) = \begin{cases} 2x, & x < 0 \\ -x^2 + 4, & x \ge 0 \end{cases}$

166. $f(x) = \begin{cases} x - 9, & x \le -1 \\ x^2 + 1, & x > -1 \end{cases}$

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3.5 Exponential and Logarithmic Models

Introduction

There are many examples of exponential and logarithmic models in real life. In Section 3.1, you used the formula

$$A = Pe^{rt} \qquad \text{Exponential model}$$

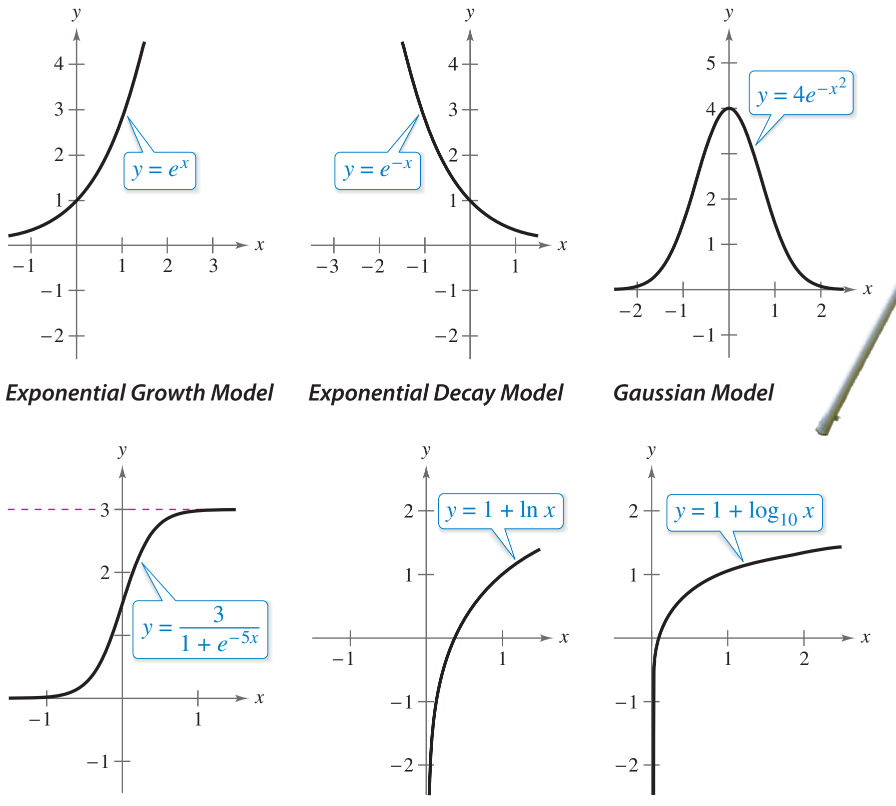
to find the balance in an account when the interest was compounded continuously. In Section 3.2, Example 10, you used the human memory model

$$f(t) = 75 - 6\ln(t + 1). \qquad \text{Logarithmic model}$$

The five most common types of mathematical models involving exponential functions or logarithmic functions are as follows.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b\ln x, \quad y = a + b\log_{10} x$

The basic shapes of these graphs are shown in Figure 3.32.



Exponential Growth Model *Exponential Decay Model* *Gaussian Model*

Logistic Growth Model *Natural Logarithmic Model* *Common Logarithmic Model*

Figure 3.32

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes.

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What you should learn

- Recognize the five most common types of models involving exponential or logarithmic functions.
- Use exponential growth and decay functions to model and solve real-life problems.
- Use Gaussian functions to model and solve real-life problems.
- Use logistic growth functions to model and solve real-life problems.
- Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Exponential decay models are used in carbon dating. For instance, in Exercise 37 on page 230, you will use an exponential decay model to estimate the age of a piece of ancient charcoal.

Archaeologist

Explore the Concept

Use a graphing utility to graph each model shown in Figure 3.32. Use the *table* and *trace* features of the graphing utility to identify the asymptotes of the graph of each function.

Exponential Growth and Decay

Example 1 Demography

Estimates of the world population (in millions) from 2003 through 2009 are shown in the table. A scatter plot of the data is shown in Figure 3.33. (Source: U.S. Census Bureau)

Year	Population, P
2003	6313
2004	6387
2005	6462
2006	6538
2007	6615
2008	6691
2009	6768

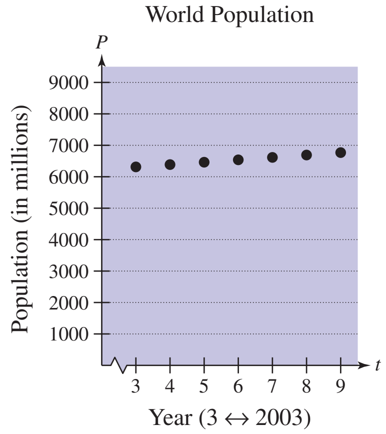


Figure 3.33

An exponential growth model that approximates these data is given by

$$P = 6097e^{0.0116t}, \quad 3 \le t \le 9$$

where P is the population (in millions) and $t = 3$ represents 2003. Compare the values given by the model with the estimates shown in the table. According to this model, when will the world population reach 7.1 billion?

Algebraic Solution

The following table compares the two sets of population figures.

Year	2003	2004	2005	2006	2007	2008	2009
Population	6313	6387	6462	6538	6615	6691	6768
Model	6313	6387	6461	6536	6613	6690	6768

From the table, it appears that the model is a good fit for the data. To find when the world population will reach 7.1 billion, let

$$P = 7100$$

in the model and solve for t.

$6097e^{0.0116t} = P$ — Write original equation.

$6097e^{0.0116t} = 7100$ — Substitute 7100 for P.

$e^{0.0116t} \approx 1.16451$ — Divide each side by 6097.

$\ln e^{0.0116t} \approx \ln 1.16451$ — Take natural log of each side.

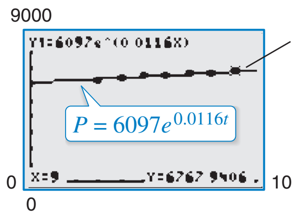
$0.0116t \approx 0.15230$ — Inverse Property

$t \approx 13.1$ — Divide each side by 0.0116.

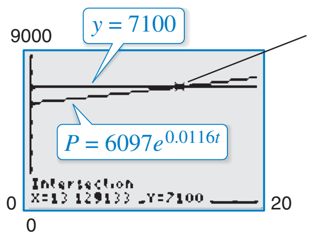
According to the model, the world population will reach 7.1 billion in 2013.

CHECKPOINT Now try Exercise 33.

Graphical Solution



The model appears to fit the data closely.



The intersection point of the model and the line $y = 7100$ is about (13.1, 7100). So, according to the model, the world population will reach 7.1 billion in 2013.

An exponential model increases (or decreases) by the same percent each year. What is the annual percent increase for the model in Example 1?

In Example 1, you were given the exponential growth model. Sometimes you must find such a model. One technique for doing this is shown in Example 2.

Example 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

Solution

Let y be the number of flies at time t (in days). From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for b, solve for a in the first equation.

$$100 = ae^{2b} \quad \Longrightarrow \quad a = \frac{100}{e^{2b}} \qquad \text{Solve for } a \text{ in the first equation.}$$

Then substitute the result into the second equation.

$$300 = ae^{4b} \qquad \text{Write second equation.}$$

$$300 = \left(\frac{100}{e^{2b}}\right)e^{4b} \qquad \text{Substitute } \frac{100}{e^{2b}} \text{ for } a.$$

$$300 = 100e^{2b} \qquad \text{Simplify.}$$

$$3 = e^{2b} \qquad \text{Divide each side by 100.}$$

$$\ln 3 = \ln e^{2b} \qquad \text{Take natural log of each side.}$$

$$\ln 3 = 2b \qquad \text{Inverse Property}$$

$$\frac{1}{2}\ln 3 = b \qquad \text{Solve for } b.$$

Using $b = \frac{1}{2}\ln 3$ and the equation you found for a, you can determine that

$$a = \frac{100}{e^{2[(1/2)\ln 3]}} \qquad \text{Substitute } \tfrac{1}{2}\ln 3 \text{ for } b.$$

$$= \frac{100}{e^{\ln 3}} \qquad \text{Simplify.}$$

$$= \frac{100}{3} \qquad \text{Inverse Property}$$

$$\approx 33.33. \qquad \text{Simplify.}$$

So, with $a \approx 33.33$ and

$$b = \tfrac{1}{2}\ln 3 \approx 0.5493$$

the exponential growth model is

$$y = 33.33e^{0.5493t},$$

as shown in Figure 3.34. This implies that after 5 days, the population will be

$$y = 33.33e^{0.5493(5)} \approx 520 \text{ flies.}$$

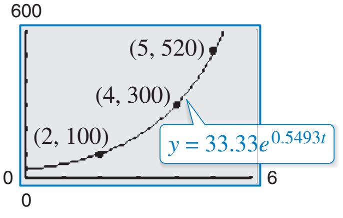


Figure 3.34

CHECKPOINT Now try Exercise 35.

In living organic material, the ratio of the content of radioactive carbon isotopes (carbon 14) to the content of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time t (in years).

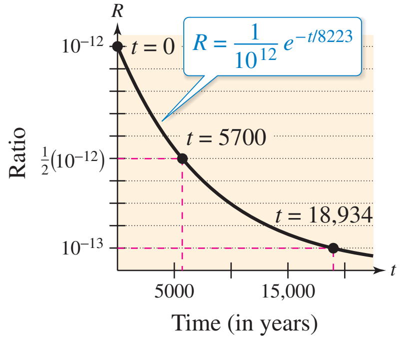


Figure 3.35

$$R = \frac{1}{10^{12}} e^{-t/8223} \qquad \text{Carbon dating model}$$

The graph of R is shown in Figure 3.35. Note that R decreases as t increases.

Example 3 Carbon Dating

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}.$$

Estimate the age of the fossil.

Algebraic Solution

In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}} e^{-t/8223} = R \qquad \text{Write original model.}$$

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Substitute } \frac{1}{10^{13}} \text{ for } R.$$

$$e^{-t/8223} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8223} = \ln \frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8223} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 18{,}934 \qquad \text{Multiply each side by } -8223.$$

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

CHECKPOINT Now try Exercise 37.

Graphical Solution

Use a graphing utility to graph the formula for the ratio of carbon 14 to carbon 12 at any time t as

$$y_1 = \frac{1}{10^{12}} e^{-x/8223}.$$

In the same viewing window, graph $y_2 = 1/(10^{13})$, as shown in Figure 3.36.

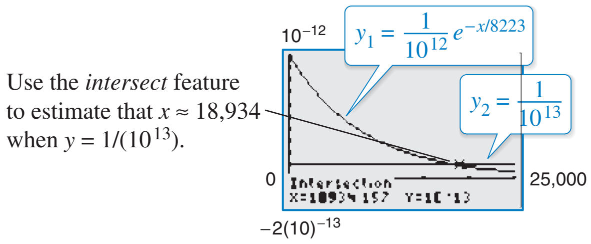


Figure 3.36

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

The carbon dating model in Example 3 assumed that the carbon 14 to carbon 12 ratio was one part in 10,000,000,000,000. Suppose an error in measurement occurred and the actual ratio was only one part in 8,000,000,000,000. The fossil age corresponding to the actual ratio would then be approximately 17,000 years. Try checking this result.

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The graph of a Gaussian model is called a **bell-shaped curve.** Try graphing the normal distribution curve with a graphing utility. Can you see why it is called a bell-shaped curve?

The average value for a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable—in this case, x.

Example 4 SAT Scores

In 2009, the Scholastic Aptitude Test (SAT) mathematics scores for college-bound seniors roughly followed the normal distribution

$$y = 0.0034e^{-(x-515)^2/26{,}912}, \quad 200 \le x \le 800$$

where x is the SAT score for mathematics. Use a graphing utility to graph this function and estimate the average SAT score. (Source: College Board)

Solution

The graph of the function is shown in Figure 3.37. On this bell-shaped curve, the maximum value of the curve represents the average score. Using the *maximum* feature of the graphing utility, you can see that the average mathematics score for college-bound seniors in 2009 was 515.

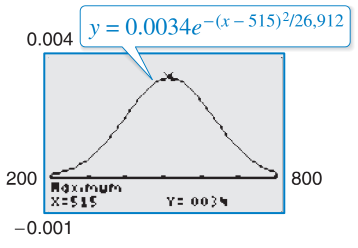


Figure 3.37

CHECKPOINT Now try Exercise 41.

In Example 4, note that 50% of the seniors who took the test received scores lower than 515 (see Figure 3.38).

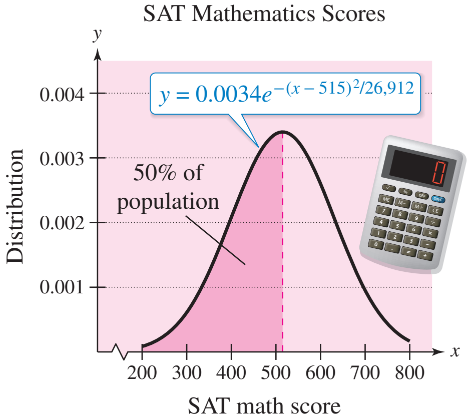


Figure 3.38

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Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 3.39. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

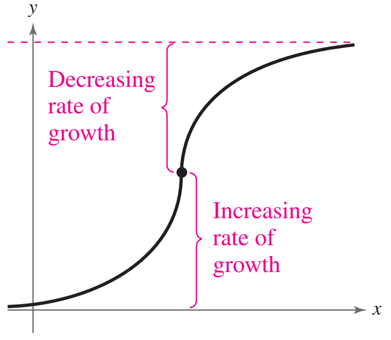


Figure 3.39 *Logistic Curve*

Example 5 Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where y is the total number of students infected after t days. The college will cancel classes when 40% or more of the students are infected.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

Algebraic Solution

a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}}$$

$$= \frac{5000}{1 + 4999e^{-4}}$$

$$\approx 54.$$

b. Classes are canceled when the number of infected students is $(0.40)(5000) = 2000$.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$\ln e^{-0.8t} = \ln \frac{1.5}{4999}$$

$$-0.8t = \ln \frac{1.5}{4999}$$

$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999}$$

$$t \approx 10.14$$

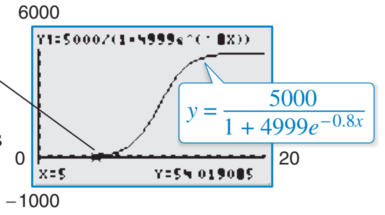
So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.

CHECKPOINT Now try Exercise 43.

Graphical Solution

a.

Use the *value* feature to estimate that $y \approx 54$ when $x = 5$. So, after 5 days, about 54 students will be infected.



b. Classes are canceled when the number of infected students is $(0.40)(5000) = 2000$. Use a graphing utility to graph

$$y_1 = \frac{5000}{1 + 4999e^{-0.8x}} \quad \text{and} \quad y_2 = 2000$$

in the same viewing window. Use the *intersect* feature of the graphing utility to find the point of intersection of the graphs, as shown in Figure 3.40.

The intersection point occurs near $x \approx 10.14$. So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.

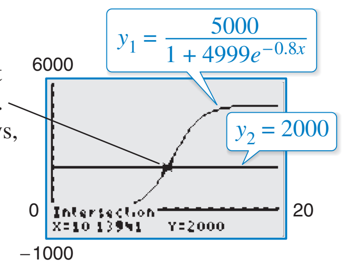


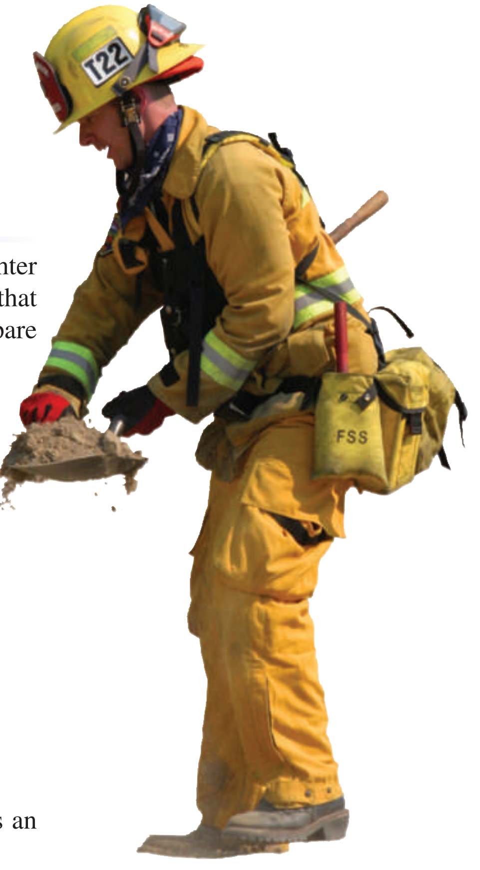
Figure 3.40

Logarithmic Models

On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Intensity is a measure of the wave energy of an earthquake.



Earthquake Relief Worker

Example 6 Magnitudes of Earthquakes

In 2009, Crete, Greece experienced an earthquake that measured 6.4 on the Richter scale. Also in 2009, the north coast of Indonesia experienced an earthquake that measured 7.6 on the Richter scale. Find the intensity of each earthquake and compare the two intensities.

Solution

Because $I_0 = 1$ and $R = 6.4$, you have

$$6.4 = \log_{10} \frac{I}{1}$$

$$10^{6.4} = 10^{\log_{10} I} \quad \Rightarrow \quad 10^{6.4} = I.$$

For $R = 7.6$, you have

$$7.6 = \log_{10} \frac{I}{1}$$

$$10^{7.6} = 10^{\log_{10} I} \quad \Rightarrow \quad 10^{7.6} = I.$$

Note that an increase of 1.2 units on the Richter scale (from 6.4 to 7.6) represents an increase in intensity by a factor of

$$\frac{10^{7.6}}{10^{6.4}} = 10^{1.2} \approx 16.$$

In other words, the intensity of the earthquake near the north coast of Indonesia was about 16 times as great as the intensity of the earthquake in Greece.

CHECKPOINT Now try Exercise 45.

Example 7 pH Levels

Acidity, or pH level, is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution. Use the model given by $\mathrm{pH} = -\log_{10}[H^+]$ to determine the hydrogen ion concentration of milk of magnesia, which has a pH of 10.5.

Solution

$$\mathrm{pH} = -\log_{10}[H^+] \qquad \text{Write original model.}$$

$$10.5 = -\log_{10}[H^+] \qquad \text{Substitute 10.5 for pH.}$$

$$-10.5 = \log_{10}[H^+] \qquad \text{Multiply each side by } -1.$$

$$10^{-10.5} = 10^{\log_{10}[H^+]} \qquad \text{Exponentiate each side (base 10).}$$

$$3.16 \times 10^{-11} = [H^+] \qquad \text{Simplify.}$$

So, the hydrogen ion concentration of milk of magnesia is 3.16×10^{-11} mole of hydrogen per liter.

CHECKPOINT Now try Exercise 51.

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3.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match the equation with its model.

(a) Exponential growth model (i) $y = ae^{-bx},\ b > 0$

(b) Exponential decay model (ii) $y = a + b \ln x$

(c) Logistic growth model (iii) $y = \dfrac{a}{1 + be^{-rx}}$

(d) Gaussian model (iv) $y = ae^{bx},\ b > 0$

(e) Natural logarithmic model (v) $y = a + b \log_{10} x$

(f) Common logarithmic model (vi) $y = ae^{-(x-b)^2/c}$

In Exercises 2 and 3, fill in the blank.

2. Gaussian models are commonly used in probability and statistics to represent populations that are _______ distributed.

3. Logistic growth curves are also called _______ curves.

4. Which model in Exercise 1 has a graph called a bell-shaped curve?

5. Does the model $y = 120e^{-0.25x}$ represent exponential growth or exponential decay?

6. Which model in Exercise 1 has a graph with two horizontal asymptotes?

Procedures and Problem Solving

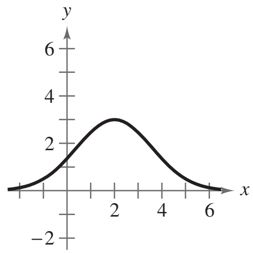
Identifying Graphs of Models **In Exercises 7–12, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

7. $y = 2e^{x/4}$ **8.** $y = 6e^{-x/4}$

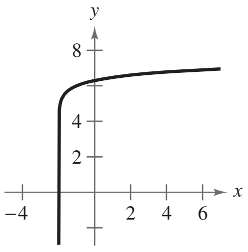
9. $y = 6 + \log_{10}(x + 2)$ **10.** $y = 3e^{-(x-2)^2/5}$

11. $y = \ln(x + 1)$ **12.** $y = \dfrac{4}{1 + e^{-2x}}$

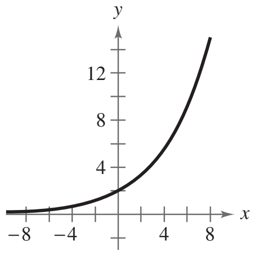
(a)



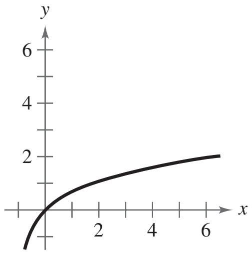
(b)



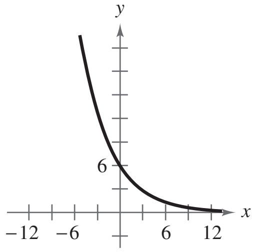
(c)



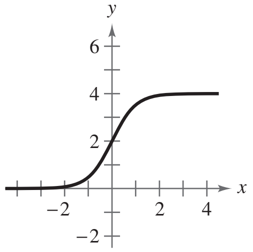
(d)



(e)



(f)



Using a Compound Interest Formula **In Exercises 13–20, complete the table for a savings account in which interest is compounded continuously.**

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
13.	$10,000	3.5%		
14.	$2000	1.5%		
15.	$7500		21 years	
16.	$1000		12 years	
17.	$5000			$5665.74
18.	$300			$385.21
19.		4.5%		$100,000.00
20.		2%		$2500.00

21. Tripling an Investment Complete the table for the time t (in years) necessary for P dollars to triple when interest is compounded continuously at rate r. Create a scatter plot of the data.

r	2%	4%	6%	8%	10%	12%
t						

22. **Tripling an Investment** Complete the table for the time t (in years) necessary for P dollars to triple when interest is compounded annually at rate r. Create a scatter plot of the data.

r	2%	4%	6%	8%	10%	12%
t						

23. **Finance** When \$1 is invested in an account over a 10-year period, the amount A in the account after t years is given by

$$A = 1 + 0.075[\![t]\!] \quad \text{or} \quad A = e^{0.07t}$$

depending on whether the account pays simple interest at $7\frac{1}{2}\%$ or continuous compound interest at 7%. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate? (Remember that $[\![t]\!]$ is the greatest integer function discussed in Section 1.3.)

24. **Finance** When \$1 is invested in an account over a 10-year period, the amount A in the account after t years is given by

$$A = 1 + 0.06[\![t]\!] \quad \text{or} \quad A = \left(1 + \frac{0.055}{365}\right)^{[\![365t]\!]}$$

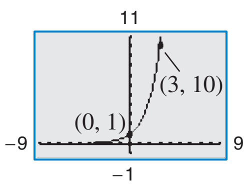
depending on whether the account pays simple interest at 6% or compound interest at $5\frac{1}{2}\%$ compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate?

Radioactive Decay **In Exercises 25–28, complete the table for the radioactive isotope.**

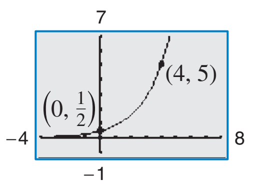
	Isotope	Half-Life (years)	Initial Quantity	Amount After 1000 Years
25.	^{226}Ra	1599	10 g	
26.	^{226}Ra	1599		1.5 g
27.	^{14}C	5700	3 g	
28.	^{239}Pu	24,100		0.4 g

Identifying a Model **In Exercises 29–32, find the exponential model $y = ae^{bx}$ that fits the points shown in the graph or table.**

29.



30.



31.

x	0	5
y	4	1

32.

x	0	3
y	1	$\frac{1}{4}$

33. **Demography** The populations P (in thousands) of Pittsburgh, Pennsylvania from 2000 through 2008 can be modeled by $P = 333.68e^{-0.0099t}$, where t is the year, with $t = 0$ corresponding to 2000. (Source: U.S. Census Bureau)

(a) According to the model, was the population of Pittsburgh increasing or decreasing from 2000 through 2008? Explain your reasoning.

(b) What were the populations of Pittsburgh in 2000, 2005, and 2008?

(c) According to the model, when will the population of Pittsburgh be approximately 290,000?

34. **MODELING DATA**

The table shows the populations (in millions) of five countries in 2005 and the projected populations (in millions) for 2015. (Source: U.S. Census Bureau)

Country	2005	2015
Australia	20.2	22.8
Canada	32.4	35.1
Hungary	10.0	9.7
Philippines	90.4	109.6
Turkey	72.7	82.5

(a) Find the exponential growth or decay model, $y = ae^{bt}$ or $y = ae^{-bt}$, for the population of each country, where t is the year, with $t = 5$ corresponding to 2005. Use the model to predict the population of each country in 2030.

(b) You can see that the populations of Canada and the Philippines are growing at different rates. What constant in the equation $y = ae^{bt}$ is determined by these different growth rates? Discuss the relationship between the different growth rates and the magnitude of the constant.

(c) The population of Turkey is increasing while the population of Hungary is decreasing. What constant in the equation $y = ae^{bt}$ reflects this difference? Explain.

35. **Demography** The populations P (in thousands) of San Antonio, Texas from 2000 through 2008 can be modeled by

$$P = 1155.4e^{kt}$$

where t is the year, with $t = 0$ corresponding to 2000. In 2002, the population was 1,200,000. (Source: U.S. Census Bureau)

(a) Find the value of k for the model. Round your result to four decimal places.

(b) Use your model to predict the population in 2015.

36. Demography The populations P (in thousands) of Raleigh, North Carolina from 2000 through 2008 can be modeled by $P = 289.81e^{kt}$, where t is the year, with $t = 0$ corresponding to 2000. In 2006, the population was 363,000. (Source: U.S. Census Bureau)

(a) Find the value of k for the model. Round your result to four decimal places.

(b) Use your model to predict the population in 2015.

37. *Why you should learn it* *(p. 221)* Carbon 14 (^{14}C) dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, then the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal given that the half-life of ^{14}C is 5700 years?

38. Radioactive Decay The half-life of radioactive radium (^{226}Ra) is 1599 years. What percent of a present amount of radioactive radium will remain after 100 years?

39. MODELING DATA

A new 2009 luxury sedan that sold for \$49,200 has a book value V of \$32,590 after 2 years.

(a) Find a linear model for the value V of the sedan.

(b) Find an exponential model for the value V of the sedan. Round the numbers in the model to four decimal places.

(c) Use a graphing utility to graph the two models in the same viewing window.

(d) Which model represents a greater depreciation rate in the first year?

(e) For what years is the value of the sedan greater using the linear model? the exponential model?

40. MODELING DATA

A new laptop computer that sold for \$935 in 2009 has a book value V of \$385 after 2 years.

(a) Find a linear model for the value V of the laptop.

(b) Find an exponential model for the value V of the laptop. Round the numbers in the model to four decimal places.

(c) Use a graphing utility to graph the two models in the same viewing window.

(d) Which model represents a greater depreciation rate in the first year?

(e) For what years is the value of the laptop greater using the linear model? the exponential model?

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41. Psychology The IQ scores for adults roughly follow the normal distribution $y = 0.0266e^{-(x-100)^2/450}$, $70 \le x \le 115$, where x is the IQ score.

(a) Use a graphing utility to graph the function.

(b) Use the graph in part (a) to estimate the average IQ score.

42. Marketing The sales S (in thousands of units) of a cleaning solution after x hundred dollars is spent on advertising are given by $S = 10(1 - e^{kx})$. When \$500 is spent on advertising, 2500 units are sold.

(a) Complete the model by solving for k.

(b) Estimate the number of units that will be sold when advertising expenditures are raised to \$700.

43. Forestry A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will follow the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where t is measured in months.

(a) What is the population after 5 months?

(b) After how many months will the population reach 500?

(c) Use a graphing utility to graph the function. Use the graph to determine the values of p at which the horizontal asymptotes occur. Identify the asymptote that is most relevant in the context of the problem and interpret its meaning.

44. Biology The number Y of yeast organisms in a culture is given by the model

$$Y = \frac{663}{1 + 72e^{-0.547t}}, \quad 0 \le t \le 18$$

where t represents the time (in hours).

(a) Use a graphing utility to graph the model.

(b) Use the model to predict the populations for the 19th hour and the 30th hour.

(c) According to this model, what is the limiting value of the population?

(d) Why do you think this population of yeast follows a logistic growth model instead of an exponential growth model?

Geology In Exercises 45 and 46, use the Richter scale (see page 227) for measuring the magnitudes of earthquakes.

45. Find the intensities I of the following earthquakes measuring R on the Richter scale (let $I_0 = 1$). (Source: U.S. Geological Survey)

(a) Haiti in 2010, $R = 7.0$

(b) Samoa Islands in 2009, $R = 8.1$

(c) Virgin Islands in 2008, $R = 6.1$

46. Find the magnitudes R of the following earthquakes of intensity I (let $I_0 = 1$).

(a) $I = 39{,}811{,}000$

(b) $I = 12{,}589{,}000$

(c) $I = 251{,}200$

Audiology **In Exercises 47–50, use the following information for determining sound intensity. The level of sound β (in decibels) with an intensity I is**

$$\beta = 10 \log_{10} \frac{I}{I_0}$$

where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 47 and 48, find the level of each sound β.

47. (a) $I = 10^{-10}$ watt per m^2 (quiet room)

(b) $I = 10^{-5}$ watt per m^2 (busy street corner)

(c) $I \approx 10^{0}$ watt per m^2 (threshold of pain)

48. (a) $I = 10^{-4}$ watt per m^2 (door slamming)

(b) $I = 10^{-3}$ watt per m^2 (loud car horn)

(c) $I = 10^{-2}$ watt per m^2 (siren at 30 meters)

49. As a result of the installation of a muffler, the noise level of an engine was reduced from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise due to the installation of the muffler.

50. As a result of the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise due to the installation of these materials.

Chemistry **In Exercises 51–54, use the acidity model given in Example 7.**

✓ **51.** Find the pH when $[H^+] = 2.3 \times 10^{-5}$.

52. Compute $[H^+]$ for a solution for which pH $= 5.8$.

53. A grape has a pH of 3.5, and baking soda has a pH of 8.0. The hydrogen ion concentration of the grape is how many times that of the baking soda?

54. The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

55. **Finance** The total interest u paid on a home mortgage of P dollars at interest rate r for t years is given by

$$u = P\left[\frac{rt}{1 - \left(\dfrac{1}{1 + r/12}\right)^{12t}} - 1\right].$$

Consider a \$120,000 home mortgage at $7\frac{1}{2}\%$.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage when the total interest paid is the same as the amount of the mortgage. Is it possible that a person could pay twice as much in interest charges as the amount of his or her mortgage?

56. **Finance** A \$120,000 home mortgage for 30 years at $7\frac{1}{2}\%$ has a monthly payment of \$839.06. Part of the monthly payment goes toward the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that goes toward the interest is given by

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is given by

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the size of the mortgage, r is the interest rate, M is the monthly payment, and t is the time (in years).

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 30 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$966.71$). What can you conclude?

57. **Forensics** At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperatures the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time (in hours elapsed since the person died) and T is the temperature (in degrees Fahrenheit) of the person's body. Assume that the person had a normal body temperature of 98.6°F at death and that the room temperature was a constant 70°F. Use the formula to estimate the time of death of the person. (This formula is derived from a general cooling principle called Newton's Law of Cooling.)

58. Culinary Arts You take a five-pound package of steaks out of a freezer at 11 A.M. and place it in a refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and the freezer temperature is 0°F. Use the formula for Newton's Law of Cooling

$$t = -5.05 \ln \frac{T - 40}{0 - 40}$$

where t is the time in hours (with $t = 0$ corresponding to 11 A.M.) and T is the temperature of the package of steaks (in degrees Fahrenheit).

Conclusions

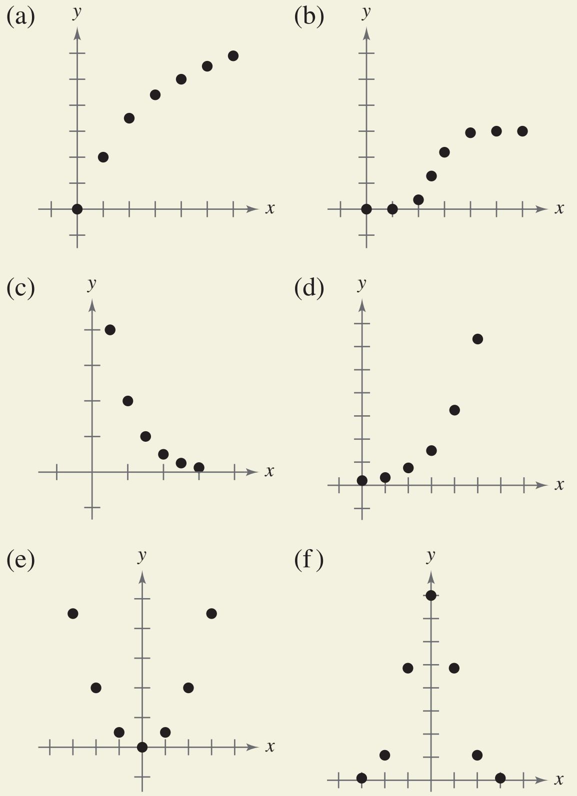
True or False? **In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.**

59. The domain of a logistic growth function cannot be the set of real numbers.

60. The graph of a logistic growth function will always have an x-intercept.

61. Think About It Can the graph of a Gaussian model ever have an x-intercept? Explain.

62. CAPSTONE For each graph, state whether an exponential, Gaussian, logarithmic, logistic, or quadratic model will fit the data best. Explain your reasoning. Then describe a real-life situation that could be represented by the data.



Identifying Models **In Exercises 63–66, identify the type of model you studied in this section that has the given characteristic.**

63. The maximum value of the function occurs at the average value of the independent variable.

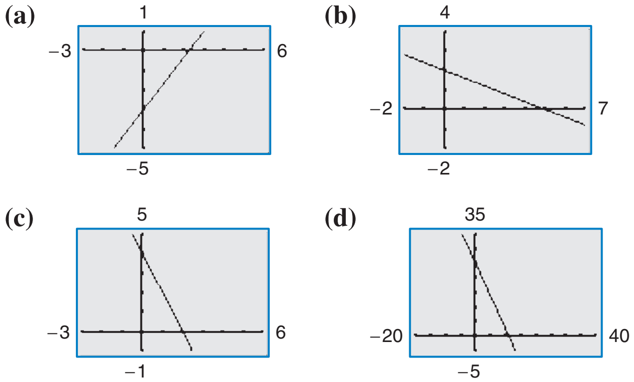
64. A horizontal asymptote of its graph represents the limiting value of a population.

65. Its graph shows a steadily increasing rate of growth.

66. The only asymptote of its graph is a vertical asymptote.

Cumulative Mixed Review

Identifying Graphs of Linear Equations **In Exercises 67–70, match the equation with its graph, and identify any intercepts. [The graphs are labeled (a), (b), (c), and (d).]**



67. $4x - 3y - 9 = 0$

68. $2x + 5y - 10 = 0$

69. $y = 25 - 2.25x$

70. $\frac{x}{2} + \frac{y}{4} = 1$

Applying the Leading Coefficient Test **In Exercises 71–74, use the Leading Coefficient Test to determine the right-hand and left-hand behavior of the graph of the polynomial function.**

71. $f(x) = 2x^3 - 3x^2 + x - 1$

72. $f(x) = 5 - x^2 - 4x^4$

73. $g(x) = -1.6x^5 + 4x^2 - 2$

74. $g(x) = 7x^6 + 9.1x^5 - 3.2x^4 + 25x^3$

Using Synthetic Division **In Exercises 75 and 76, divide using synthetic division.**

75. $(2x^3 - 8x^2 + 3x - 9) \div (x - 4)$

76. $(x^4 - 3x + 1) \div (x + 5)$

77. *Make a Decision* To work an extended application analyzing the net sales for Kohl's Corporation from 1992 through 2008, visit this textbook's *Companion Website*. (Data Source: Kohl's Illinois, Inc.)

3.6 Nonlinear Models

Classifying Scatter Plots

In Section 1.7, you saw how to fit linear models to data, and in Section 2.8, you saw how to fit quadratic models to data. In real life, many relationships between two variables are represented by different types of growth patterns. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

What you should learn

- Classify scatter plots.
- Use scatter plots and a graphing utility to find models for data and choose the model that best fits a set of data.
- Use a graphing utility to find exponential and logistic models for data.

Why you should learn it

Many real-life applications can be modeled by nonlinear equations. For instance, in Exercise 34 on page 240, you are asked to find a nonlinear model that relates air pressure to altitude.

Example 1 Classifying Scatter Plots

Decide whether each set of data could best be modeled by a linear model, $y = ax + b$, an exponential model, $y = ab^x$, or a logarithmic model, $y = a + b \ln x$.

a. (2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5), (4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5), (6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5), (8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)

b. (2, 2), (2.5, 3.1), (3, 3.8), (3.5, 4.3), (4, 4.6), (4.5, 5.3), (5, 5.6), (5.5, 5.9), (6, 6.2), (6.5, 6.4), (7, 6.9), (7.5, 7.2), (8, 7.6), (8.5, 7.9), (9, 8), (9.5, 8.2)

c. (2, 1.9), (2.5, 2.5), (3, 3.2), (3.5, 3.6), (4, 4.3), (4.5, 4.7), (5, 5.2), (5.5, 5.7), (6, 6.4), (6.5, 6.8), (7, 7.2), (7.5, 7.9), (8, 8.6), (8.5, 8.9), (9, 9.5), (9.5, 9.9)

Solution

a. From Figure 3.41, it appears that the data can best be modeled by an exponential function.

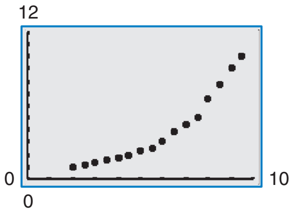


Figure 3.41

b. From Figure 3.42, it appears that the data can best be modeled by a logarithmic function.

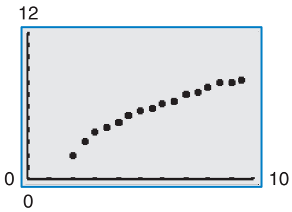


Figure 3.42

c. From Figure 3.43, it appears that the data can best be modeled by a linear function.

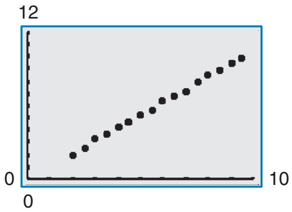


Figure 3.43

CHECKPOINT Now try Exercise 13.

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Fitting Nonlinear Models to Data

Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

Example 2 Fitting a Model to Data

Fit the following data from Example 1(a) to an exponential model and a power model. Identify the coefficient of determination and determine which model fits the data better.

(2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5),

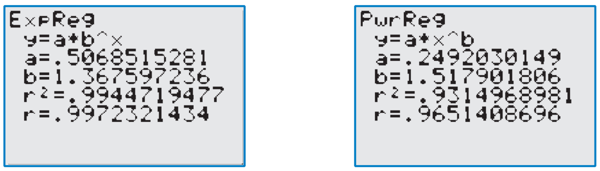
(4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5),

(6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5),

(8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)

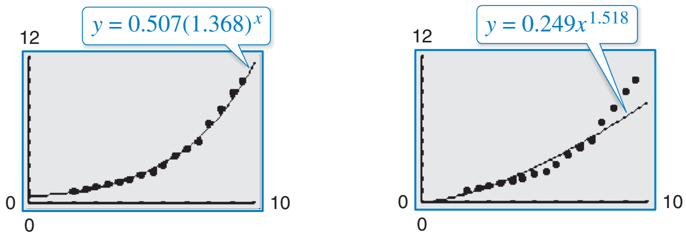
Solution

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find exponential and power models for the data, as shown in Figure 3.44.



Exponential Model **Power Model**
Figure 3.44

So, an exponential model for the data is $y = 0.507(1.368)^x$, and a power model for the data is $y = 0.249x^{1.518}$. Plot the data and each model in the same viewing window, as shown in Figure 3.45. To determine which model fits the data better, compare the coefficients of determination for each model. The model whose r^2-value is closest to 1 is the model that better fits the data. In this case, the better-fitting model is the exponential model.



Exponential Model **Power Model**
Figure 3.45

CHECKPOINT Now try Exercise 31.

Deciding which model best fits a set of data is a question that is studied in detail in statistics. Recall from Section 1.7 that the model that best fits a set of data is the one whose *sum of squared differences* is the least. In Example 2, the sums of squared differences are 0.90 for the exponential model and 14.30 for the power model.

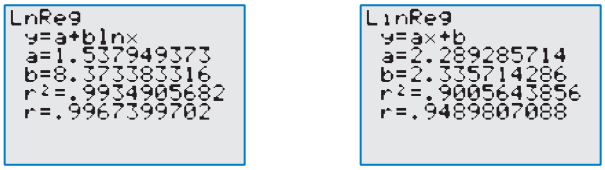
Example 3 Fitting a Model to Data

The table shows the yield y (in milligrams) of a chemical reaction after x minutes. Use a graphing utility to find a logarithmic model and a linear model for the data and identify the coefficient of determination for each model. Determine which model fits the data better.

Minutes, x	Yield, y
1	1.5
2	7.4
3	10.2
4	13.4
5	15.8
6	16.3
7	18.2
8	18.3

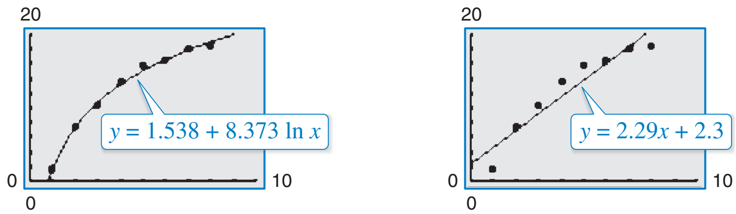
Solution

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find logarithmic and linear models for the data, as shown in Figure 3.46.



Logarithmic Model **Linear Model**
Figure 3.46

So, a logarithmic model for the data is $y = 1.538 + 8.373 \ln x$ and a linear model for the data is $y = 2.29x + 2.3$. Plot the data and each model in the same viewing window, as shown in Figure 3.47. To determine which model fits the data better, compare the coefficients of determination for each model. The model whose coefficient of determination is closer to 1 is the model that better fits the data. In this case, the better-fitting model is the logarithmic model.



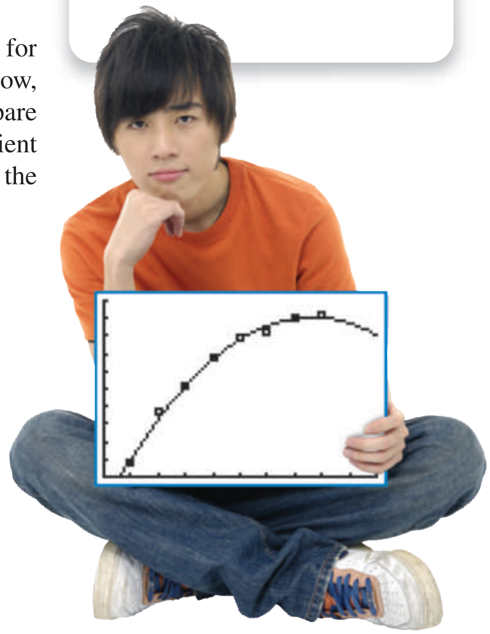
Logarithmic Model **Linear Model**
Figure 3.47

CHECKPOINT Now try Exercise 33.

In Example 3, the sum of the squared differences for the logarithmic model is 1.59 and the sum of the squared differences for the linear model is 24.31.

Explore the Concept

Use a graphing utility to find a quadratic model for the data in Example 3. Do you think this model fits the data better than the logarithmic model in Example 3? Explain your reasoning.



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Modeling With Exponential and Logistic Functions

Example 4 Fitting an Exponential Model to Data

The table at the right shows the amounts of revenue R (in billions of dollars) collected by the Internal Revenue Service (IRS) for selected years from 1963 through 2008. Use a graphing utility to find a model for the data. Then use the model to estimate the revenue collected in 2013. (Source: IRS Data Book)

Year	Revenue, R
1963	105.9
1968	153.6
1973	237.8
1978	399.8
1983	627.2
1988	935.1
1993	1176.7
1998	1769.4
2003	1952.9
2008	2745.0

Solution

Let x represent the year, with $x = 3$ corresponding to 1963. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 3.48.

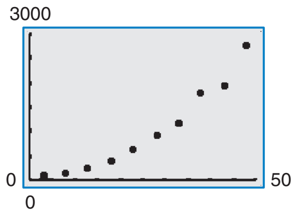


Figure 3.48

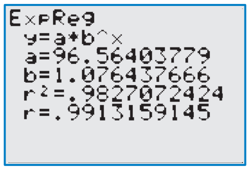


Figure 3.49

From the scatter plot, it appears that an exponential model is a good fit. Use the *regression* feature of the graphing utility to find the exponential model, as shown in Figure 3.49. Change the model to a natural exponential model, as follows.

$$R = 96.56(1.076)^x \quad \text{Write original model.}$$

$$= 96.56e^{(\ln 1.076)x} \quad b = e^{\ln b}$$

$$\approx 96.56e^{0.073x} \quad \text{Simplify.}$$

Graph the data and the natural exponential model

$$R = 96.56e^{0.073x}$$

in the same viewing window, as shown in Figure 3.50. From the model, you can see that the revenue collected by the IRS from 1963 through 2008 had an average annual increase of about 7%. From this model, you can estimate the 2013 revenue to be

$$R = 96.56e^{0.073x} \quad \text{Write natural exponential model.}$$

$$= 96.56e^{0.073(53)} \quad \text{Substitute 53 for } x.$$

$$\approx \$4624.7 \text{ billion} \quad \text{Use a calculator.}$$

which is more than twice the amount collected in 2003. You can also use the *value* feature of the graphing utility to approximate the revenue in 2013 to be \$4624.7 billion, as shown in Figure 3.50.

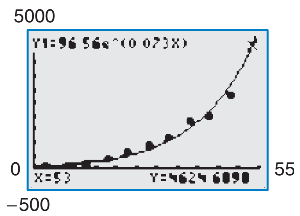


Figure 3.50

CHECKPOINT Now try Exercise 35.

Study Tip

You can change an exponential model of the form

$$y = ab^x$$

to one of the form

$$y = ae^{cx}$$

by rewriting b in the form

$$b = e^{\ln b}.$$

For instance,

$$y = 3(2^x)$$

can be written as

$$y = 3(2^x) = 3e^{(\ln 2)x} \approx 3e^{0.693x}.$$

The next example demonstrates how to use a graphing utility to fit a logistic model to data.

Example 5 Fitting a Logistic Model to Data

To estimate the amount of defoliation caused by the gypsy moth during a given year, a forester counts the number x of egg masses on $\frac{1}{40}$ of an acre (circle of radius 18.6 feet) in the fall. The percent of defoliation y the next spring is shown in the table. (Source: USDA, Forest Service)

Egg masses, x	Percent of defoliation, y
0	12
25	44
50	81
75	96
100	99

Forester

a. Use the *regression* feature of a graphing utility to find a logistic model for the data.

b. How closely does the model represent the data?

Graphical Solution

a. Enter the data into a graphing utility. Using the *regression* feature of the graphing utility, you can find the logistic model, as shown in Figure 3.51.

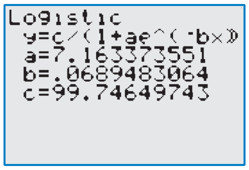


Figure 3.51

You can approximate this model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}.$$

b. You can use the graphing utility to graph the actual data and the model in the same viewing window. In Figure 3.52, it appears that the model is a good fit for the actual data.

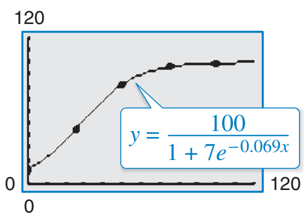


Figure 3.52

CHECKPOINT Now try Exercise 37.

Numerical Solution

a. Enter the data into a graphing utility. Using the *regression* feature of the graphing utility, you can approximate the logistic model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}.$$

b. You can see how well the model fits the data by comparing the actual values of y with the values of y given by the model, which are labeled y^* in the table below.

x	0	25	50	75	100
y	12	44	81	96	99
y^*	12.5	44.5	81.8	96.2	99.3

In the table, you can see that the model appears to be a good fit for the actual data.

3.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. A power model has the form _______ .

2. An exponential model of the form $y = ab^x$ can be rewritten as a natural exponential model of the form _______.

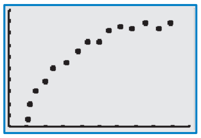
3. What type of visual display can you create to get an idea of which type of model will best fit the data set?

4. A power model for a set of data has a coefficient of determination of $r^2 \approx 0.901$ and an exponential model for the data has a coefficient of determination of $r^2 \approx 0.967$. Which model fits the data better?

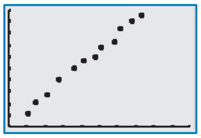
Procedures and Problem Solving

Classifying Scatter Plots **In Exercises 5–12, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, an exponential model, a logarithmic model, or a logistic model.**

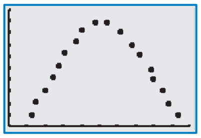
5.



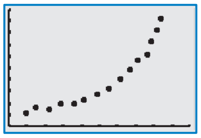
6.



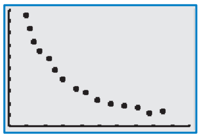
7.



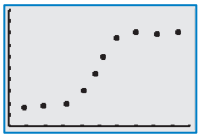
8.



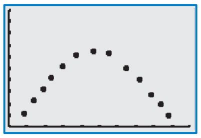
9.



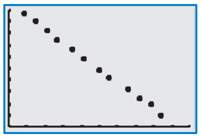
10.



11.



12.



Classifying Scatter Plots **In Exercises 13–18, use a graphing utility to create a scatter plot of the data. Decide whether the data could best be modeled by a linear model, an exponential model, or a logarithmic model.**

13. (1, 2.0), (1.5, 3.5), (2, 4.0), (4, 5.8), (6, 7.0), (8, 7.8)

14. (1, 5.8), (1.5, 6.0), (2, 6.5), (4, 7.6), (6, 8.9), (8, 10.0)

15. (1, 4.4), (1.5, 4.7), (2, 5.5), (4, 9.9), (6, 18.1), (8, 33.0)

16. (1, 11.0), (1.5, 9.6), (2, 8.2), (4, 4.5), (6, 2.5), (8, 1.4)

17. (1, 7.5), (1.5, 7.0), (2, 6.8), (4, 5.0), (6, 3.5), (8, 2.0)

18. (1, 5.0), (1.5, 6.0), (2, 6.4), (4, 7.8), (6, 8.6), (8, 9.0)

Finding an Exponential Model **In Exercises 19–22, use the *regression* feature of a graphing utility to find an exponential model $y = ab^x$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.**

19. (0, 5), (1, 6), (2, 7), (3, 9), (4, 13)

20. (0, 4.0), (2, 6.9), (4, 18.0), (6, 32.3), (8, 59.1), (10, 118.5)

21. (0, 10.0), (1, 6.1), (2, 4.2), (3, 3.8), (4, 3.6)

22. (−3, 120.2), (0, 80.5), (3, 64.8), (6, 58.2), (10, 55.0)

Finding a Logarithmic Model **In Exercises 23–26, use the *regression* feature of a graphing utility to find a logarithmic model $y = a + b \ln x$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.**

23. (1, 2.0), (2, 3.0), (3, 3.5), (4, 4.0), (5, 4.1), (6, 4.2), (7, 4.5)

24. (1, 8.5), (2, 11.4), (4, 12.8), (6, 13.6), (8, 14.2), (10, 14.6)

25. (1, 10), (2, 6), (3, 6), (4, 5), (5, 3), (6, 2)

26. (3, 14.6), (6, 11.0), (9, 9.0), (12, 7.6), (15, 6.5)

Finding a Power Model **In Exercises 27–30, use the *regression* feature of a graphing utility to find a power model $y = ax^b$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.**

27. (1, 2.0), (2, 3.4), (5, 6.7), (6, 7.3), (10, 12.0)

28. (0.5, 1.0), (2, 12.5), (4, 33.2), (6, 65.7), (8, 98.5), (10, 150.0)

29. (1, 10.0), (2, 4.0), (3, 0.7), (4, 0.1)

30. (2, 450), (4, 385), (6, 345), (8, 332), (10, 312)

✓ 31. MODELING DATA

The table shows the yearly sales S (in millions of dollars) of Whole Foods Market for the years 2001 through 2008. (Source: Whole Foods Market)

Year	Sales, S
2001	2272.2
2002	2690.5
2003	3148.6
2004	3865.0
2005	4701.3
2006	5607.4
2007	6591.8
2008	7953.9

(a) Use the *regression* feature of a graphing utility to find an exponential model and a power model for the data and identify the coefficient of determination for each model. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the graphing utility to graph each model with the data.

(c) Use the coefficients of determination to determine which model best fits the data.

32. MODELING DATA

The table shows the annual amounts A (in billions of dollars) spent in the U.S. by the cruise lines and passengers of the North American cruise industry from 2003 through 2008. (Source: Cruise Lines International Association)

Year	Amount, A
2003	12.92
2004	14.70
2005	16.18
2006	17.64
2007	18.70
2008	19.07

(a) Use the *regression* feature of a graphing utility to find a linear model, an exponential model, and a logarithmic model for the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the graphing utility to graph each model with the data. Use the graphs to determine which model best fits the data.

(c) Use the model you chose in part (b) to predict the amount spent in 2009. Is the amount reasonable?

✓ 33. MODELING DATA

The populations P (in millions) of the United States for the years 1995 through 2008 are shown in the table, where t represents the year, with $t = 5$ corresponding to 1995. (Source: U.S. Census Bureau)

Year	Population, P
1995	266.6
1996	269.7
1997	272.9
1998	276.1
1999	279.3
2000	282.4
2001	285.3
2002	288.0
2003	290.7
2004	293.3
2005	296.0
2006	298.8
2007	301.7
2008	304.5

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(b) Use the *regression* feature of the graphing utility to find a power model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(c) Use the *regression* feature of the graphing utility to find an exponential model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(d) Use the *regression* feature of the graphing utility to find a logarithmic model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(e) Which model is the best fit for the data? Explain.

(f) Use each model to predict the populations of the United States for the years 2009 through 2014.

(g) Which model is the best choice for predicting the future population of the United States? Explain.

(h) Were your choices of models the same for parts (e) and (g)? If not, explain why your choices were different.

34. ***Why you should learn it*** *(p. 233)* The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is approximately 1.03323 kilograms per square centimeter, and this pressure is called one atmosphere. Variations in weather conditions cause changes in the atmospheric pressure of up to ± 5 percent. The ordered pairs (h, p) give the pressures p (in atmospheres) for various altitudes h (in kilometers).

$(0, 1), (10, 0.25), (20, 0.06), (5, 0.55),$
$(15, 0.12), (25, 0.02)$

(a) Use the *regression* feature of a graphing utility to attempt to find the logarithmic model $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use the *regression* feature of the graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use the graphing utility to plot the data and graph the logarithmic model in the same viewing window.

(d) Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.

(e) Use the graph in part (c) to estimate the pressure at an altitude of 13 kilometers.

✓ **35. MODELING DATA**

The table shows the numbers N of office supply stores operated by Staples from 2001 through 2008. (Source: Staples, Inc.)

Year	Number, N
2001	1436
2002	1488
2003	1559
2004	1680
2005	1780
2006	1884
2007	2038
2008	2218

(a) Use the *regression* feature of a graphing utility to find an exponential model for the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Rewrite the model as a natural exponential model.

(c) Use the natural exponential model to predict the number of Staples stores in 2009. Is the number reasonable?

36. MODELING DATA

A beaker of liquid at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the liquid is measured every 5 minutes for a period of $\frac{1}{2}$ hour. The results are recorded in the table, where t is the time (in minutes) and T is the temperature (in degrees Celsius).

Time, t	Temperature, T
0	78.0°
5	66.0°
10	57.5°
15	51.2°
20	46.3°
25	42.5°
30	39.6°

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Do the data appear linear? Explain.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Do the data appear quadratic? Even though the quadratic model appears to be a good fit, explain why it might not be a good model for predicting the temperature of the liquid when $t = 60$.

(c) The graph of the temperature of the room should be an asymptote of the graph of the model. Subtract the room temperature from each of the temperatures in the table. Use the *regression* feature of the graphing utility to find an exponential model for the revised data. Add the room temperature to this model. Use the graphing utility to plot the original data and graph the model in the same viewing window.

(d) Explain why the procedure in part (c) was necessary for finding the exponential model.

✓ **37. MODELING DATA**

The table shows the percents P of women in different age groups (in years) who have been married at least once. (Source: U.S. Census Bureau)

Age group	Percent, P
20–24	21.1
25–29	54.5
30–34	73.9
35–39	84.0
40–44	86.5
45–54	89.7
55–64	93.1
65–74	95.8

(a) Use the *regression* feature of a graphing utility to find a logistic model for the data. Let x represent the midpoint of the age group.

(b) Use the graphing utility to graph the model with the original data. How closely does the model represent the data?

38. MODELING DATA

The table shows the annual sales S (in millions of dollars) of AutoZone for the years from 2002 through 2009. (Source: AutoZone, Inc.)

Year	Sales, S
2002	5325.5
2003	5457.1
2004	5637.0
2005	5710.9
2006	5948.4
2007	6169.8
2008	6522.7
2009	6816.8

(a) Use the *regression* feature of a graphing utility to find a logarithmic model, an exponential model, and a power model for the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) Use each of the following methods to choose the model that best fits the data. Compare your results.

(i) Create a table of values for each model.

(ii) Use the graphing utility to graph each model with the data.

(iii) Find and compare the coefficients of determination for the models.

Conclusions

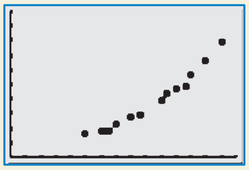
True or False? **In Exercises 39 and 40, determine whether the statement is true or false. Justify your answer.**

39. The exponential model $y = ae^{bx}$ represents a growth model when $b > 0$.

40. To change an exponential model of the form $y = ab^x$ to one of the form $y = ae^{cx}$, rewrite b as $b = \ln e^b$.

41. Writing In your own words, explain how to fit a model to a set of data using a graphing utility.

42. CAPSTONE You use a graphing utility to create the scatter plot of a set of data.



(a) What types of models are likely to fit the data well? Explain.

(b) Discuss the methods you can use to find the model of best fit for the data. Which method would you prefer? Explain.

Cumulative Mixed Review

Using the Slope-Intercept Form **In Exercises 43–46, find the slope and y-intercept of the equation of the line. Then sketch the line by hand.**

43. $2x + 5y = 10$

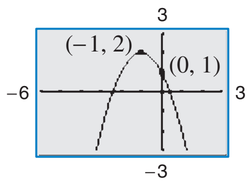
44. $3x - 2y = 9$

45. $1.2x + 3.5y = 10.5$

46. $0.4x - 2.5y = 12.0$

Writing the Equation of a Parabola in Standard Form **In Exercises 47–50, write an equation of the parabola in standard form.**

47.



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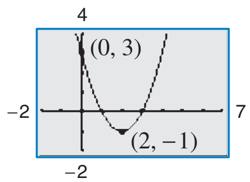

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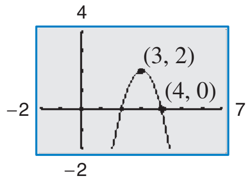

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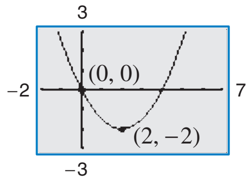
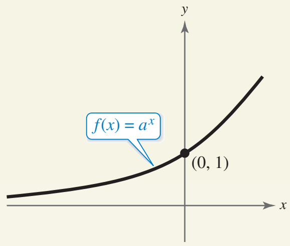
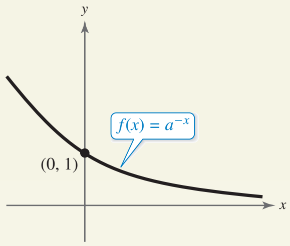
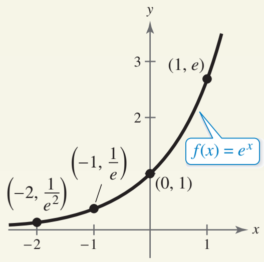
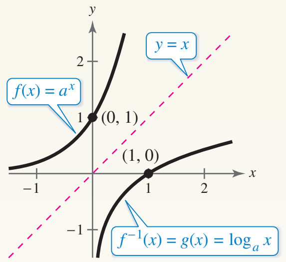
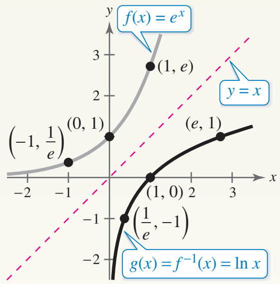


3 Chapter Summary

<table>
<tr><th></th><th>What did you learn?</th><th>Explanation and Examples</th><th>Review Exercises</th></tr>
<tr><td rowspan="4">3.1</td><td>Recognize and evaluate exponential functions with base a (p. 180).</td><td>The exponential function f with base a is denoted by $f(x) = a^x$, where $a > 0$, $a \neq 1$, and x is any real number.</td><td>1–4</td></tr>
<tr><td>Graph exponential functions with base a (p. 181).</td><td>


</td><td>5–12</td></tr>
<tr><td>Recognize, evaluate, and graph exponential functions with base e (p. 184).</td><td>The function $f(x) = e^x$ is called the natural exponential function.

</td><td>13–18</td></tr>
<tr><td>Use exponential functions to model and solve real-life problems (p. 186).</td><td>Exponential functions are used in compound interest formulas (see Example 8) and in radioactive decay models (see Example 10).</td><td>19–22</td></tr>
<tr><td rowspan="3">3.2</td><td>Recognize and evaluate logarithmic functions with base a (p. 192).</td><td>For $x > 0$, $a > 0$, and $a \neq 1$, $y = \log_a x$ if and only if $x = a^y$. The function $f(x) = \log_a x$ is called the logarithmic function with base a.</td><td>23–36</td></tr>
<tr><td>Graph logarithmic functions with base a (p. 194), and recognize, evaluate, and graph natural logarithmic functions (p. 196).</td><td>The graphs of $g(x) = \log_a x$ and $f(x) = a^x$ are reflections of each other in the line $y = x$.


The function defined by $g(x) = \ln x$, $x > 0$, is called the natural logarithmic function.

</td><td>37–48</td></tr>
<tr><td>Use logarithmic functions to model and solve real-life problems (p. 198).</td><td>A logarithmic function is used in the human memory model. (See Example 10.)</td><td>49, 50</td></tr>
<tr><td>3.3</td><td>Rewrite logarithms with different bases (p. 203).</td><td>Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.
Base b: $\log_a x = \dfrac{\log_b x}{\log_b a}$
Base 10: $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$
Base e: $\log_a x = \dfrac{\ln x}{\ln a}$</td><td>51–58</td></tr>
</table>

	What did you learn?	Explanation and Examples	Review Exercises
3.3	**Use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions *(p. 204)*.**	Let a be a positive number ($a \neq 1$), n be a real number, and u and v be positive real numbers. **1. Product Property:** $\log_a(uv) = \log_a u + \log_a v$ $\ln(uv) = \ln u + \ln v$ **2. Quotient Property:** $\log_a(u/v) = \log_a u - \log_a v$ $\ln(u/v) = \ln u - \ln v$ **3. Power Property:** $\log_a u^n = n \log_a u$, $\ln u^n = n \ln u$	59–78
	Use logarithmic functions to model and solve real-life problems *(p. 206)*.	Logarithmic functions can be used to find an equation that relates the periods of several planets and their distances from the sun. (See Example 7.)	79, 80
3.4	**Solve simple exponential and logarithmic equations *(p. 210)*.**	Solve simple exponential or logarithmic equations using the One-to-One Properties and Inverse Properties of exponential and logarithmic functions.	81–94
	Solve more complicated exponential *(p. 211)* and logarithmic *(p. 213)* equations.	To solve more complicated equations, rewrite the equations so that the One-to-One Properties and Inverse Properties of exponential and logarithmic functions can be used. (See Examples 2–8.)	95–118
	Use exponential and logarithmic equations to model and solve real-life problems *(p. 216)*.	Exponential and logarithmic equations can be used to find how long it will take to double an investment (see Example 12) and to find the year in which the average salary for public school teachers reached $45,000 (see Example 13).	119, 120
3.5	**Recognize the five most common types of models involving exponential or logarithmic functions *(p. 221)*.**	**1. Exponential growth model:** $y = ae^{bx}$, $b > 0$ **2. Exponential decay model:** $y = ae^{-bx}$, $b > 0$ **3. Gaussian model:** $y = ae^{-(x-b)^2/c}$ **4. Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$ **5. Logarithmic models:** $y = a + b \ln x$ $y = a + b \log_{10} x$	121–126
	Use exponential growth and decay functions to model and solve real-life problems *(p. 222)*.	An exponential growth function can be used to model the world population (see Example 1) and an exponential decay function can be used to estimate the age of a fossil (see Example 3).	127
	Use Gaussian functions *(p. 225)*, logistic growth functions *(p. 226)*, and logarithmic functions *(p. 227)* to model and solve real-life problems.	A Gaussian function can be used to model SAT math scores for college-bound seniors (see Example 4). A logistic growth function can be used to model the spread of a flu virus (see Example 5). A logarithmic function can be used to find the intensity of an earthquake using its magnitude (see Example 6).	128–130
3.6	**Classify scatter plots *(p. 233)*, and use scatter plots and a graphing utility to find models for data and choose the model that best fits a set of data *(p. 234)*.**	You can use a scatter plot and a graphing utility to choose a model that best fits a set of data that represents the yield of a chemical reaction. (See Example 3.)	131, 132
	Use a graphing utility to find exponential and logistic models for data *(p. 236)*.	An exponential model can be used to estimate the amount of revenue collected by the Internal Revenue Service for a given year (see Example 4) and a logistic model can be used to estimate the percent of defoliation caused by the gypsy moth (see Example 5).	133, 134

3 Review Exercises

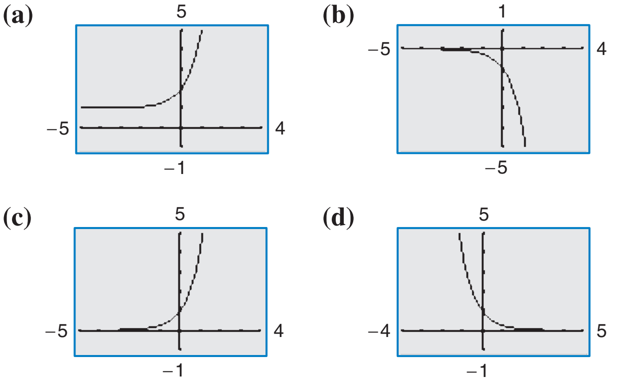
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

3.1

Evaluating Exponential Functions **In Exercises 1–4, use a calculator to evaluate the function at the indicated value of x. Round your result to four decimal places.**

1. $f(x) = 1.45^x, \quad x = 2\pi$
2. $f(x) = 7^x, \quad x = -\sqrt{11}$
3. $g(x) = 60^{2x}, \quad x = -1.1$
4. $g(x) = 25^{-3x}, \quad x = \frac{3}{2}$

Library of Parent Functions **In Exercises 5–8, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**



5. $f(x) = 4^x$
6. $f(x) = 4^{-x}$
7. $f(x) = -4^x$
8. $f(x) = 4^x + 1$

Graphs of $y = a^x$ and $y = a^{-x}$ **In Exercises 9–12, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.**

9. $f(x) = 6^x$
10. $f(x) = 0.3^x$
11. $g(x) = 6^{-x}$
12. $g(x) = 0.3^{-x}$

Graphing an Exponential Function **In Exercises 13–18, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.**

13. $h(x) = e^{x-1}$
14. $f(x) = e^{x+2}$
15. $h(x) = -e^x$
16. $f(x) = 3 - e^{-x}$
17. $f(x) = 4e^{-0.5x}$
18. $f(x) = 2 + e^{x+3}$

Finding the Balance for Compound Interest **In Exercises 19 and 20, complete the table to determine the balance A for \$10,000 invested at rate r for t years, compounded continuously.**

t	1	10	20	30	40	50
A						

19. $r = 8\%$
20. $r = 3\%$

21. **Economics** A new SUV costs \$32,000. The value V of the SUV after t years is modeled by $V(t) = 32{,}000\left(\frac{3}{4}\right)^t$.
 (a) Use a graphing utility to graph the function.
 (b) Find the value of the SUV after 2 years.
 (c) According to the model, when does the SUV depreciate most rapidly? Is this realistic? Explain.

22. **Radioactive Decay** Let Q represent the mass, in grams, of a quantity of plutonium 241 (^{241}Pu), whose half-life is 14 years. The quantity of plutonium present after t years is given by $Q = 50\left(\frac{1}{2}\right)^{t/14}$.
 (a) Determine the initial quantity (when $t = 0$).
 (b) Determine the quantity present after 10 years.
 (c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 50$.

3.2

Rewriting Equations **In Exercises 23–32, write the logarithmic equation in exponential form or write the exponential equation in logarithmic form.**

23. $\log_5 125 = 3$
24. $\log_9 81 = 2$
25. $\log_{64} 2 = \frac{1}{6}$
26. $\log_{10}\left(\frac{1}{100}\right) = -2$
27. $4^3 = 64$
28. $3^5 = 243$
29. $125^{2/3} = 25$
30. $12^{-1} = \frac{1}{12}$
31. $\left(\frac{1}{2}\right)^{-3} = 8$
32. $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$

Evaluating Logarithms **In Exercises 33–36, use the definition of logarithmic function to evaluate the function at the indicated value of x without using a calculator.**

	Function	*Value*
33.	$f(x) = \log_6 x$	$x = 216$
34.	$f(x) = \log_7 x$	$x = 1$
35.	$f(x) = \log_4 x$	$x = \frac{1}{4}$
36.	$f(x) = \log_{10} x$	$x = 0.00001$

Sketching the Graph of a Logarithmic Function **In Exercises 37–40, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand.**

37. $g(x) = -\log_2 x + 5$
38. $g(x) = \log_5(x - 3)$
39. $f(x) = \log_2(x - 1) + 6$
40. $f(x) = \log_5(x + 2) - 3$

Evaluating the Natural Logarithmic Function **In Exercises 41–44, use a calculator to evaluate the function $f(x) = \ln x$ at the indicated value of x. Round your result to three decimal places, if necessary.**

41. $x = 21.5$
42. $x = 0.46$
43. $x = \sqrt{6}$
44. $x = \frac{5}{6}$

Analyzing Graphs of Functions **In Exercises 45–48, use a graphing utility to graph the logarithmic function. Determine the domain and identify any vertical asymptote and x-intercept.**

45. $f(x) = \ln x + 3$
46. $f(x) = \ln(x - 3)$
47. $h(x) = \frac{1}{2}\ln x$
48. $f(x) = \frac{1}{4}\ln x$

49. Aeronautics The time t (in minutes) for a small plane to climb to an altitude of h feet is given by

$$t = 50 \log_{10}[18{,}000/(18{,}000 - h)]$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the function and identify any asymptotes.

(c) As the plane approaches its absolute ceiling, what can be said about the time required to further increase its altitude?

(d) Find the amount of time it will take for the plane to climb to an altitude of 4000 feet.

50. Real Estate The model

$$t = 12.542 \ln[x/(x - 1000)], \quad x > 1000$$

approximates the length of a home mortgage of \$150,000 at 8% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.

(a) Use the model to approximate the length of a \$150,000 mortgage at 8% when the monthly payment is \$1254.68.

(b) Approximate the total amount paid over the term of the mortgage with a monthly payment of \$1254.68. What amount of the total is interest costs?

3.3

Changing the Base **In Exercises 51–54, evaluate the logarithm using the change-of-base formula. Do each problem twice, once with common logarithms and once with natural logarithms. Round your results to three decimal places.**

51. $\log_4 9$
52. $\log_{1/2} 9$
53. $\log_{14} 364$
54. $\log_3 0.28$

Graphing a Logarithm with Any Base **In Exercises 55–58, use the change-of-base formula and a graphing utility to graph the function.**

55. $f(x) = \log_2(x - 1)$
56. $f(x) = 2 - \log_3 x$
57. $f(x) = -\log_{1/2}(x + 2)$
58. $f(x) = \log_{1/3}(x - 1) + 1$

Using Properties to Evaluate Logarithms **In Exercises 59–62, approximate the logarithm using the properties of logarithms, given the values $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.**

59. $\log_b 9$
60. $\log_b \frac{4}{9}$
61. $\log_b \sqrt{5}$
62. $\log_b 50$

Simplifying a Logarithm **In Exercises 63–66, use the properties of logarithms to rewrite and simplify the logarithmic expression.**

63. $\ln(5e^{-2})$
64. $\ln \sqrt{e^5}$
65. $\log_{10} 200$
66. $\log_{10} 0.002$

Expanding Logarithmic Expressions **In Exercises 67–72, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

67. $\log_5 5x^2$
68. $\log_4 16xy^2$
69. $\log_{10} \dfrac{5\sqrt{y}}{x^2}$
70. $\ln \dfrac{\sqrt{x}}{4}$
71. $\ln \dfrac{x + 3}{xy}$
72. $\ln \dfrac{xy^5}{\sqrt{z}}$

Condensing Logarithmic Expressions **In Exercises 73–78, condense the expression to the logarithm of a single quantity.**

73. $\log_2 9 + \log_2 x$
74. $\log_6 y - 2\log_6 z$
75. $\frac{1}{2}\ln(2x - 1) - 2\ln(x + 1)$
76. $5\ln(x - 2) - \ln(x + 2) - 3\ln x$
77. $\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$
78. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

79. Public Service The number of miles s of roads cleared of snow in 1 hour is approximated by the model

$$s = 25 - \frac{13\ln(h/12)}{\ln 3}, \quad 2 \le h \le 15$$

where h is the depth of the snow (in inches).

(a) Use a graphing utility to graph the function.

(b) Complete the table.

h	4	6	8	10	12	14
s						

(c) Using the graph of the function and the table, what conclusion can you make about the number of miles of roads cleared as the depth of the snow increases?

80. Psychology Students in a sociology class were given an exam and then retested monthly with an equivalent exam. The average scores for the class are given by the human memory model $f(t) = 85 - 17\log_{10}(t + 1)$, where t is the time in months and $0 \le t \le 10$. When will the average score decrease to 68?

3.4

Solving an Exponential or Logarithmic Equation **In Exercises 81–94, solve the equation for x without using a calculator.**

81. $10^x = 10{,}000$ **82.** $7^x = 343$

83. $6^x = \frac{1}{216}$ **84.** $6^{x-2} = 1296$

85. $2^{x+1} = \frac{1}{16}$ **86.** $4^{x/2} = 64$

87. $\log_8 x = 4$ **88.** $\log_x 729 = 6$

89. $\log_2(x - 1) = 3$ **90.** $\log_5(2x + 1) = 2$

91. $\ln x = 4$ **92.** $\ln x = -3$

93. $\ln(x - 1) = 2$ **94.** $\ln(2x + 1) = -4$

Solving an Exponential Equation **In Exercises 95–104, solve the exponential equation algebraically. Round your result to three decimal places.**

95. $3e^{-5x} = 132$ **96.** $14e^{3x+2} = 560$

97. $2^x + 13 = 35$ **98.** $6^x - 28 = -8$

99. $-4(5^x) = -68$ **100.** $2(12^x) = 190$

101. $2e^{x-3} - 1 = 4$ **102.** $-e^{x/2} + 1 = \frac{1}{2}$

103. $e^{2x} - 7e^x + 10 = 0$ **104.** $e^{2x} - 6e^x + 8 = 0$

Solving a Logarithmic Equation **In Exercises 105–114, solve the logarithmic equation algebraically. Round your result to three decimal places.**

105. $\ln 3x = 6.4$ **106.** $\ln 5x = 4.5$

107. $\ln x - \ln 5 = 2$ **108.** $\ln x - \ln 3 = 4$

109. $\ln\sqrt{x + 1} = 2$ **110.** $\ln\sqrt{x + 40} = 3$

111. $\log_4(x - 1) = \log_4(x - 2) - \log_4(x + 2)$

112. $\log_5(x + 2) - \log_5 x = \log_5(x + 5)$

113. $\log_{10}(1 - x) = -1$

114. $\log_{10}(-x - 4) = 2$

Solving an Exponential or Logarithmic Equation **In Exercises 115–118, solve the equation algebraically. Round your result to three decimal places.**

115. $xe^x + e^x = 0$ **116.** $2xe^{2x} + e^{2x} = 0$

117. $x\ln x + x = 0$ **118.** $\dfrac{1 - \ln x}{x^2} = 0$

119. Finance You deposit \$7550 in an account that pays 6.9% interest, compounded continuously. How long will it take for the money to double?

120. Economics The demand x for a 32-inch plasma television is modeled by

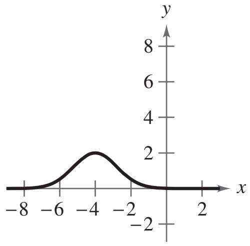
$$p = 5000\left(1 - \frac{4}{4 + e^{-0.0005x}}\right).$$

Find the demands x for prices of (a) $p = \$450$ and (b) $p = \$400$.

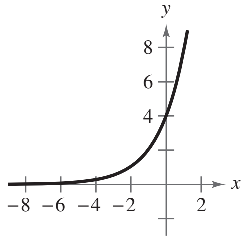
3.5

Identifying Graphs of Models **In Exercises 121–126, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

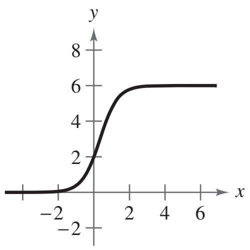
(a)



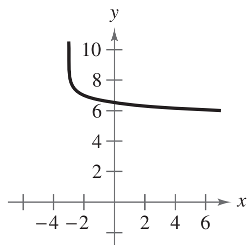
(b)



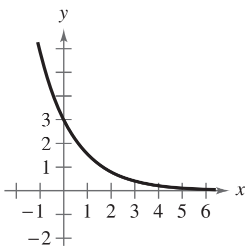
(c)



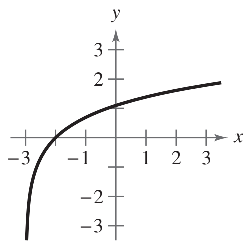
(d)



(e)



(f)



121. $y = 3e^{-2x/3}$ **122.** $y = 4e^{2x/3}$

123. $y = \ln(x + 3)$ **124.** $y = 7 - \log_{10}(x + 3)$

125. $y = 2e^{-(x+4)^2/3}$ **126.** $y = \dfrac{6}{1 + 2e^{-2x}}$

127. Demography The populations P (in thousands) of North Carolina from 1990 through 2008 can be modeled by $P = 6707.7e^{kt}$, where t is the year, with $t = 0$ corresponding to 1990. In 2008, the population was about 9,222,000. Find the value of k and use the result to predict the population in the year 2020. (Source: U.S. Census Bureau)

128. Education The scores for a biology test follow a normal distribution modeled by $y = 0.0499e^{-(x-74)^2/128}$, where x is the test score and $40 \le x \le 100$.

(a) Use a graphing utility to graph the function.

(b) Use the graph to estimate the average test score.

129. Education The average number N of words per minute that the students in a first grade class could read orally after t weeks of school is modeled by

$$N = \frac{62}{1 + 5.4e^{-0.24t}}.$$

Find the numbers of weeks it took the class to read at average rates of (a) 40 words per minute and (b) 60 words per minute.

130. Geology On the Richter scale, the magnitude R of an earthquake of intensity I is modeled by

$$R = \log_{10} \frac{I}{I_0}$$

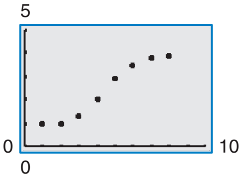
where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensities I of the following earthquakes measuring R on the Richter scale.

(a) $R = 7.1$ (b) $R = 8.4$ (c) $R = 5.5$

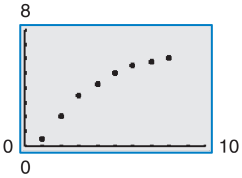
3.6

Classifying Scatter Plots In Exercises 131 and 132, determine whether the scatter plot could best be modeled by a linear model, an exponential model, a logarithmic model, or a logistic model.

131.



132.



133. MODELING DATA

Each ordered pair (t, N) represents the year t and the number N (in thousands) of female participants in high school athletic programs during nine school years, with $t = 1$ corresponding to the 2000–2001 school year. (Source: National Federation of State High School Associations)

(1, 2784), (2, 2807), (3, 2856), (4, 2865), (5, 2908), (6, 2953), (7, 3022), (8, 3057), (9, 3114)

(a) Use the *regression* feature of a graphing utility to find a linear model, an exponential model, and a power model for the data and identify the coefficient of determination for each model.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Explain.

(d) Use the model you chose in part (c) to predict the number of participants during the 2009–2010 school year.

(e) Use the model you chose in part (c) to predict the school year in which about 3,720,000 girls will participate.

134. MODELING DATA

You plant a tree when it is 1 meter tall and check its height h (in meters) every 10 years, as shown in the table.

Year	Height, h
0	1
10	3
20	7.5
30	14.5
40	19
50	20.5
60	21

(a) Use the *regression* feature of a graphing utility to find a logistic model for the data. Let x represent the year.

(b) Use the graphing utility to graph the model with the original data.

(c) How closely does the model represent the data?

(d) What is the limiting height of the tree?

Conclusions

True or False? In Exercises 135–138, determine whether the equation or statement is true or false. Justify your answer.

135. $e^{x-1} = \dfrac{e^x}{e}$

136. $\ln(x + y) = \ln(xy)$

137. The domain of the function $f(x) = \ln x$ is the set of all real numbers.

138. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

139. Think About It Without using a calculator, explain why you know that $2^{\sqrt{2}}$ is greater than 2, but less than 4.

140. Exploration

(a) Use a graphing utility to compare the graph of the function $y = e^x$ with the graph of each function below. [$n!$ (read as "n factorial") is defined as $n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n$.]

$$y_1 = 1 + \frac{x}{1!}, \quad y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!},$$

$$y_3 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

(b) Identify the pattern of successive polynomials given in part (a). Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$. What do you think this pattern implies?

3 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, use a graphing utility to construct a table of values for the function. Then sketch a graph of the function. Identify any asymptotes and intercepts.

1. $f(x) = 10^{-x}$ **2.** $f(x) = -6^{x-2}$ **3.** $f(x) = 1 - e^{2x}$

In Exercises 4–6, evaluate the expression.

4. $\log_7 7^{-0.89}$ **5.** $4.6 \ln e^2$ **6.** $5 - \log_{10} 1000$

In Exercises 7–9, find the domain, vertical asymptote, and *x*-intercept of the logarithmic function, and sketch its graph by hand.

7. $f(x) = -\log_{10} x - 6$ **8.** $f(x) = \ln(x - 4)$ **9.** $f(x) = 1 + \ln(x + 6)$

In Exercises 10–12, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

10. $\log_7 44$ **11.** $\log_{2/5} 0.9$ **12.** $\log_{12} 64$

In Exercises 13–15, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms.

13. $\log_2 3a^4$ **14.** $\ln \dfrac{5\sqrt{x}}{6}$ **15.** $\ln \dfrac{x\sqrt{x+1}}{2e^4}$

In Exercises 16–18, condense the expression to the logarithm of a single quantity.

16. $\log_3 13 + \log_3 y$ **17.** $4 \ln x - 4 \ln y$

18. $\ln x - \ln(x + 2) + \ln(2x - 3)$

In Exercises 19–22, solve the equation for *x*.

19. $3^x = 81$ **20.** $5^{2x} = 2500$

21. $\log_7 x = 3$ **22.** $\log_{10}(x - 4) = 5$

In Exercises 23–26, solve the equation algebraically. Round your result to three decimal places.

23. $\dfrac{1025}{8 + e^{4x}} = 5$ **24.** $-xe^{-x} + e^{-x} = 0$

25. $\log_{10} x - \log_{10}(8 - 5x) = 2$ **26.** $2x \ln x - x = 0$

27. The half-life of radioactive actinium (^{227}Ac) is 22 years. What percent of a present amount of radioactive actinium will remain after 19 years?

28. The table shows the annual revenues R (in millions of dollars) for Daktronics from 2001 through 2008. (Source: Daktronics, Inc.)

(a) Use the *regression* feature of a graphing utility to find a logarithmic model, an exponential model, and a power model for the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Use the model to predict the revenue of Daktronics in 2015.

Year	Revenue, R
2001	148.8
2002	177.8
2003	209.9
2004	230.3
2005	309.4
2006	433.2
2007	499.7
2008	581.9

1–3 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test to review the material in Chapters 1–3. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) write the slope-intercept form of the equation of the line that satisfies the given conditions and (b) find three additional points through which the line passes.

1. The line contains the points $(-5, 8)$ and $(-1, 4)$.

2. The line contains the point $\left(-\frac{1}{2}, 1\right)$ and has a slope of -2.

3. The line has an undefined slope and contains the point $\left(-\frac{3}{7}, \frac{1}{8}\right)$.

In Exercises 4 and 5, evaluate the function at each value of the independent variable and simplify.

4. $f(x) = \dfrac{x}{x-2}$

(a) $f(5)$ (b) $f(2)$ (c) $f(5+4s)$

5. $f(x) = \begin{cases} 3x - 8, & x < 0 \\ x^2 + 4, & x \geq 0 \end{cases}$

(a) $f(-8)$ (b) $f(0)$ (c) $f(4)$

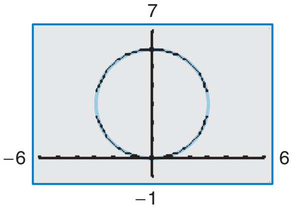


Figure for 6

6. Does the graph at the right represent y as a function of x? Explain.

7. Use a graphing utility to graph the function $f(x) = 2|x - 5| - |x + 5|$. Then determine the open intervals over which the function is increasing, decreasing, or constant.

8. Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

(a) $r(x) = \dfrac{1}{4}\sqrt{x}$ (b) $h(x) = \sqrt{x} - 3$ (c) $g(x) = -\sqrt{x+3}$

In Exercises 9–12, evaluate the indicated function for

$$f(x) = -x^2 + 3x - 10 \quad \text{and} \quad g(x) = 4x + 1.$$

9. $(f + g)(-4)$

10. $(g - f)\left(\frac{3}{4}\right)$

11. $(g \circ f)(-2)$

12. $(fg)(-1)$

13. Determine whether $h(x) = 5x - 2$ has an inverse function. If so, find it.

In Exercises 14–16, sketch the graph of the function. Use a graphing utility to verify the graph.

14. $f(x) = -(x - 2)^2 + 5$

15. $f(x) = x^2 - 6x + 5$

16. $f(x) = x^3 + 2x^2 - 9x - 18$

17. Find all the zeros of $f(x) = x^3 + 2x^2 + 4x + 8$.

18. Use a graphing utility to approximate any real zeros of $g(x) = x^3 + 4x^2 - 11$ accurate to three decimal places.

19. Divide $(4x^2 + 14x - 9)$ by $(x + 3)$ using long division.

20. Divide $(2x^3 - 5x^2 + 6x - 20)$ by $(x - 6)$ using synthetic division.

21. Multiply the complex number $-5 + 4i$ by its complex conjugate.

22. Find a polynomial function with real coefficients that has the zeros 0, -3, and $1 + \sqrt{5}i$.

In Exercises 23–25, sketch the graph of the rational function. Identify any asymptotes. Use a graphing utility to verify your graph.

23. $f(x) = \dfrac{2x}{x-3}$ **24.** $f(x) = \dfrac{5x}{x^2+x-6}$ **25.** $f(x) = \dfrac{x^2-3x+8}{x-2}$

In Exercises 26–29, use a calculator to evaluate the expression. Round your answer to three decimal places.

26. $(1.85)^{3.1}$ **27.** $58^{\sqrt{5}}$ **28.** $e^{-8/5}$ **29.** $4e^{2.56}$

In Exercises 30–33, sketch the graph of the function by hand. Use a graphing utility to verify your graph.

30. $f(x) = -3^{x+4} - 5$ **31.** $f(x) = -\left(\frac{1}{2}\right)^{-x} - 3$

32. $f(x) = 4 + \log_{10}(x-3)$ **33.** $f(x) = \ln(4-x)$

In Exercises 34–36, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

34. $\log_5 16$ **35.** $\log_9 6.8$ **36.** $\log_2\left(\frac{3}{2}\right)$

37. Use the properties of logarithms to expand $\ln \dfrac{x^2-4}{x^2+1}$.

38. Write $2 \ln x - \ln(x-1) + \ln(x+1)$ as a logarithm of a single quantity.

In Exercises 39–44, solve the equation algebraically. Round your result to three decimal places and verify your result graphically.

39. $6e^{2x} = 72$

40. $4^{x-5} + 21 = 30$

41. $\log_2 x + \log_2 5 = 6$

42. $250e^{0.05x} = 500{,}000$

43. $2x^2e^{2x} - 2xe^{2x} = 0$

44. $\ln(2x-5) - \ln x = 1$

45. A rectangular plot of land with a perimeter of 546 feet has a width of x.

(a) Write the area A of the plot as a function of x.

(b) Use a graphing utility to graph the area function. What is the domain of the function?

(c) Approximate the dimensions of the plot when the area is 15,000 square feet.

46. The table shows the average prices y (in dollars) received by commercial trout producers per pound of trout in the United States from 2001 to 2008. (Source: U.S. Department of Agriculture)

(a) Use the *regression* feature of a graphing utility to find a quadratic model, an exponential model, and a power model for the data and identify the coefficient of determination for each model. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Explain.

(d) Use the model you chose in part (c) to predict the average price of one pound of trout in 2010. Is your answer reasonable? Explain.

Year	Average price, y (in dollars)
2001	1.13
2002	1.08
2003	1.04
2004	1.03
2005	1.05
2006	1.11
2007	1.19
2008	1.38

Proofs in Mathematics

Each of the following three properties of logarithms can be proved by using properties of exponential functions.

Properties of Logarithms (p. 204)

Let a be a positive real number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	*Natural Logarithm*
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Slide Rules

The slide rule was invented by William Oughtred (1574–1660) in 1625. The slide rule is a computational device with a sliding portion and a fixed portion. A slide rule enables you to perform multiplication by using the Product Property of logarithms. There are other slide rules that allow for the calculation of roots and trigonometric functions. Slide rules were used by mathematicians and engineers until the invention of the handheld calculator in 1972.

Proof

Let

$$x = \log_a u \quad \text{and} \quad y = \log_a v.$$

The corresponding exponential forms of these two equations are

$$a^x = u \quad \text{and} \quad a^y = v.$$

To prove the Product Property, multiply u and v to obtain

$$uv = a^x a^y = a^{x+y}.$$

The corresponding logarithmic form of $uv = a^{x+y}$ is

$$\log_a(uv) = x + y.$$

So,

$$\log_a(uv) = \log_a u + \log_a v.$$

To prove the Quotient Property, divide u by v to obtain

$$\frac{u}{v} = \frac{a^x}{a^y} = a^{x-y}.$$

The corresponding logarithmic form of $u/v = a^{x-y}$ is

$$\log_a \frac{u}{v} = x - y.$$

So,

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

To prove the Power Property, substitute a^x for u in the expression $\log_a u^n$, as follows.

$$\begin{aligned} \log_a u^n &= \log_a (a^x)^n && \text{Substitute } a^x \text{ for } u. \\ &= \log_a a^{nx} && \text{Property of exponents} \\ &= nx && \text{Inverse Property of logarithms} \\ &= n \log_a u && \text{Substitute } \log_a u \text{ for } x. \end{aligned}$$

So, $\log_a u^n = n \log_a u$.

Progressive Summary (Chapters 1–3)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, and 9. In each Progressive Summary, new topics encountered for the first time appear in red.

ALGEBRAIC FUNCTIONS

Polynomial, Rational, Radical

Rewriting

Polynomial form ↔ Factored form
Operations with polynomials
Rationalize denominators
Simplify rational expressions
Operations with complex numbers

Solving

Equation	*Strategy*
Linear	Isolate variable
Quadratic	Factor, set to zero Extract square roots Complete the square Quadratic Formula
Polynomial	Factor, set to zero Rational Zero Test
Rational	Multiply by LCD
Radical	Isolate, raise to power
Absolute value	Isolate, form two equations

Analyzing

Graphically

Intercepts
Symmetry
Slope
Asymptotes
End behavior
Minimum values
Maximum values

Algebraically

Domain, Range
Transformations
Composition
Standard forms of equations
Leading Coefficient Test
Synthetic division
Descartes's Rule of Signs

Numerically

Table of values

TRANSCENDENTAL FUNCTIONS

Exponential, Logarithmic

Rewriting

Exponential form ↔ Logarithmic form
Condense/expand logarithmic expressions

Solving

Equation	*Strategy*
Exponential	Take logarithm of each side
Logarithmic	Exponentiate each side

Analyzing

Graphically

Intercepts
Asymptotes

Algebraically

Domain, Range
Transformations
Composition
Inverse Properties

Numerically

Table of values

OTHER TOPICS

Rewriting

Solving

Analyzing