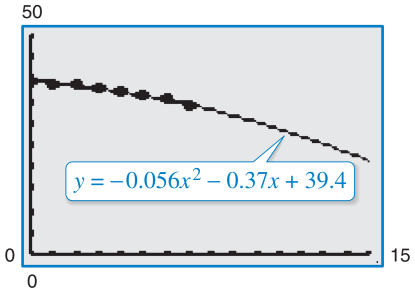
2 Polynomial and Rational Functions



Section 2.8, Example 4
Regular Soft Drinks Consumed

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2.1 Quadratic Functions

The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions.

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x of degree n.**

Polynomial functions are classified by degree. For instance, the polynomial function

$$f(x) = a, \quad a \neq 0 \qquad \text{Constant function}$$

has degree 0 and is called a **constant function.** In Chapter 1, you learned that the graph of this type of function is a horizontal line. The polynomial function

$$f(x) = mx + b, \quad m \neq 0 \qquad \text{Linear function}$$

has degree 1 and is called a **linear function.** You learned in Chapter 1 that the graph of $f(x) = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

Definition of Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function given by

$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

is called a **quadratic function.**

Often real-life data can be modeled by quadratic functions. For instance, the table at the right shows the height h (in feet) of a projectile fired from a height of 6 feet with an initial velocity of 256 feet per second at any time t (in seconds). A quadratic model for the data in the table is

$$h(t) = -16t^2 + 256t + 6 \quad \text{for} \quad 0 \leq t \leq 16.$$

t	h
0	6
2	454
4	774
6	966
8	1030
10	966
12	774
14	454
16	6

The graph of a quadratic function is a special type of U-shaped curve called a **parabola.** Parabolas occur in many real-life applications, especially those involving reflective properties, such as satellite dishes or flashlight reflectors. You will study these properties in a later chapter.

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is called the **vertex** of the parabola.

What you should learn

- Analyze graphs of quadratic functions.
- Write quadratic functions in standard form and use the results to sketch graphs of functions.
- Find minimum and maximum values of quadratic functions in real-life applications.

Why you should learn it

Quadratic functions can be used to model the design of a room. For instance, Exercise 63 on page 97 shows how the size of an indoor fitness room with a running track can be modeled.

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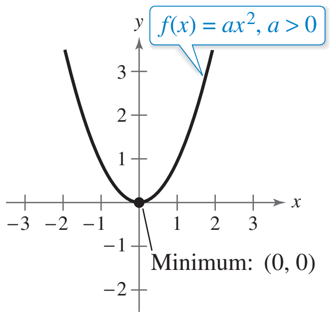
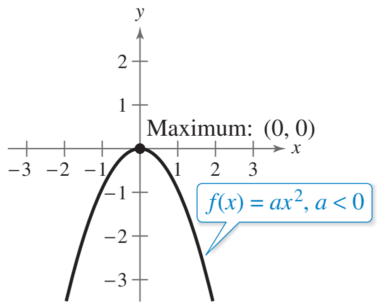
Basic Characteristics of Quadratic Functions

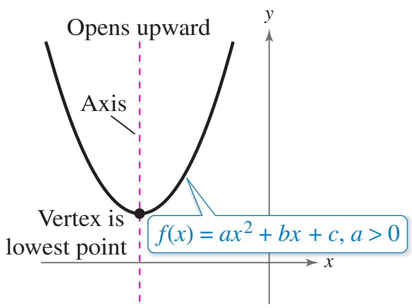
Graph of $f(x) = ax^2, \ a > 0$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative minimum or vertex: $(0, 0)$

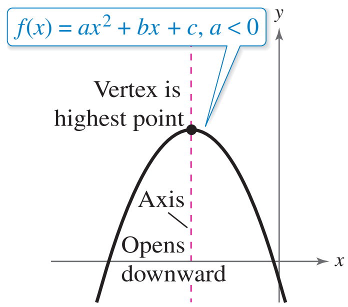
Graph of $f(x) = ax^2, \ a < 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 0]$
Intercept: $(0, 0)$
Increasing on $(-\infty, 0)$
Decreasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative maximum or vertex: $(0, 0)$



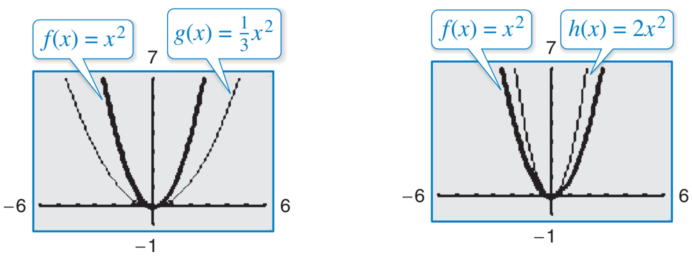


For the general quadratic form $f(x) = ax^2 + bx + c$, when the leading coefficient a is positive, the parabola opens upward; and when the leading coefficient a is negative, the parabola opens downward. Later in this section you will learn ways to find the coordinates of the vertex of a parabola.





When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 1.4. There you saw that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. Notice in Figure 2.1 that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. When $|a|$ is small, the parabola opens more widely than when $|a|$ is large.



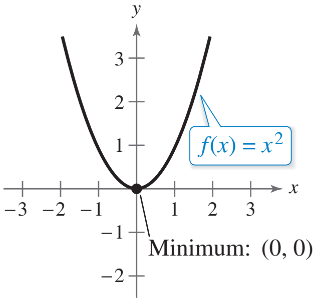
Vertical shrink *Vertical stretch*

Figure 2.1

Library of Parent Functions: Quadratic Function

The *parent quadratic function* is $f(x) = x^2$, also known as the *squaring function*. The basic characteristics of the parent quadratic function are summarized below and on the inside cover of this text.

Graph of $f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative minimum or vertex: $(0, 0)$



Example 1 Library of Parent Functions: $f(x) = x^2$

Sketch the graph of the function and describe how the graph is related to the graph of $f(x) = x^2$.

a. $g(x) = -x^2 + 1$

b. $h(x) = (x + 2)^2 - 3$

Solution

a. With respect to the graph of $f(x) = x^2$, the graph of g is obtained by a *reflection* in the x-axis and a vertical shift one unit *upward*, as shown in Figure 2.2. Confirm this with a graphing utility.

b. With respect to the graph of $f(x) = x^2$, the graph of h is obtained by a horizontal shift two units *to the left* and a vertical shift three units *downward*, as shown in Figure 2.3. Confirm this with a graphing utility.

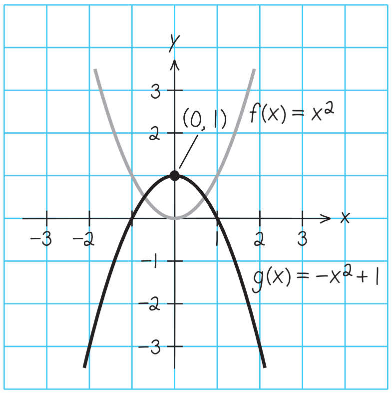


Figure 2.2

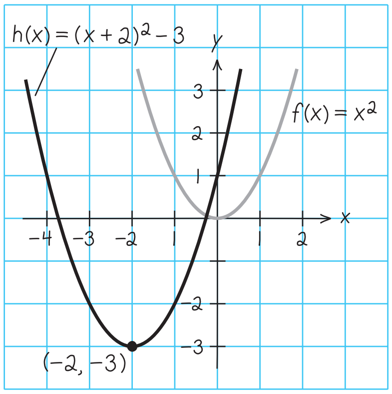


Figure 2.3

CHECKPOINT Now try Exercise 11.

Recall from Section 1.4 that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = -f(x)$, and $y = f(-x)$ are rigid transformations of the graph of $y = f(x)$.

$y = f(x \pm c)$	Horizontal shift	$y = -f(x)$	Reflection in x-axis
$y = f(x) \pm c$	Vertical shift	$y = f(-x)$	Reflection in y-axis

The Standard Form of a Quadratic Function

The equation in Example 1(b) is written in the **standard form**

$$f(x) = a(x - h)^2 + k.$$

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). When $a > 0$, the parabola opens upward, and when $a < 0$, the parabola opens downward.

Explore the Concept

Use a graphing utility to graph $y = ax^2$ with $a = -2, -1, -0.5, 0.5, 1,$ and 2. How does changing the value of a affect the graph?

Use a graphing utility to graph $y = (x - h)^2$ with $h = -4, -2, 2,$ and 4. How does changing the value of h affect the graph?

Use a graphing utility to graph $y = x^2 + k$ with $k = -4, -2, 2,$ and 4. How does changing the value of k affect the graph?

Example 2 Identifying the Vertex of a Quadratic Function

Describe the graph of

$$f(x) = 2x^2 + 8x + 7$$

and identify the vertex.

Solution

Write the quadratic function in standard form by completing the square. Recall that the first step is to factor out any coefficient of x^2 that is not 1.

$$f(x) = 2x^2 + 8x + 7 \qquad \text{Write original function.}$$

$$= (2x^2 + 8x) + 7 \qquad \text{Group } x\text{-terms.}$$

$$= 2(x^2 + 4x) + 7 \qquad \text{Factor 2 out of } x\text{-terms.}$$

$$= 2(x^2 + 4x + \underbrace{4}_{\left(\frac{4}{2}\right)^2} - 4) + 7 \qquad \text{Add and subtract } (4/2)^2 = 4 \text{ within parentheses to complete the square.}$$

$$= 2(x^2 + 4x + 4) - 2(4) + 7 \qquad \text{Regroup terms.}$$

$$= 2(x + 2)^2 - 1 \qquad \text{Write in standard form.}$$

From the standard form, you can see that the graph of f is a parabola that opens upward with vertex

$$(-2, -1)$$

as shown in Figure 2.4. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of

$$y = 2x^2.$$

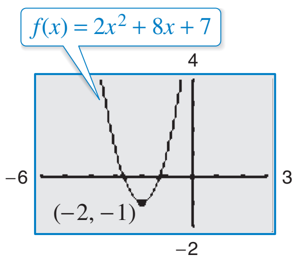


Figure 2.4

CHECKPOINT Now try Exercise 23.

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, solve the equation $ax^2 + bx + c = 0$. When $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts, or a graphing utility to approximate the x-intercepts. Remember, however, that a parabola may not have x-intercepts.

Example 3 Identifying x-Intercepts of a Quadratic Function

Describe the graph of $f(x) = -x^2 + 6x - 8$ and identify any x-intercepts.

Solution

$f(x) = -x^2 + 6x - 8$ Write original function.

$= -(x^2 - 6x) - 8$ Factor -1 out of x-terms.

$= -(x^2 - 6x + 9 - 9) - 8$ Because $b = 6$, add and subtract $(6/2)^2 = 9$ within parentheses.

$\left(\frac{6}{2}\right)^2$

$= -(x^2 - 6x + 9) - (-9) - 8$ Regroup terms.

$= -(x - 3)^2 + 1$ Write in standard form.

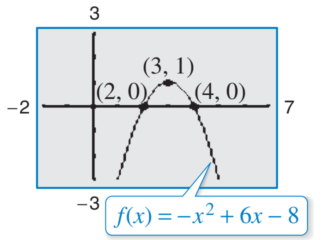


Figure 2.5

The graph of f is a parabola that opens downward with vertex $(3, 1)$, as shown in Figure 2.5. The x-intercepts are determined as follows.

$-(x^2 - 6x + 8) = 0$ Factor out -1.

$-(x - 2)(x - 4) = 0$ Factor.

$x - 2 = 0 \Rightarrow x = 2$ Set 1st factor equal to 0.

$x - 4 = 0 \Rightarrow x = 4$ Set 2nd factor equal to 0.

So, the x-intercepts are $(2, 0)$ and $(4, 0)$, as shown in Figure 2.5.

CHECKPOINT Now try Exercise 31.

Example 4 Writing the Equation of a Parabola in Standard Form

Write the standard form of the equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(3, -6)$.

Solution

Because the vertex of the parabola is $(h, k) = (1, 2)$, the equation has the form

$f(x) = a(x - 1)^2 + 2.$ Substitute for h and k in standard form.

Because the parabola passes through the point $(3, -6)$, it follows that $f(3) = -6$. So, you obtain

$f(x) = a(x - 1)^2 + 2$ Write in standard form.

$-6 = a(3 - 1)^2 + 2$ Substitute -6 for $f(x)$ and 3 for x.

$-6 = 4a + 2$ Simplify.

$-8 = 4a$ Subtract 2 from each side.

$-2 = a.$ Divide each side by 4.

The equation in standard form is

$$f(x) = -2(x - 1)^2 + 2.$$

You can confirm this answer by graphing $f(x) = -2(x - 1)^2 + 2$ with a graphing utility, as shown in Figure 2.6. Use the *zoom* and *trace* features or the *maximum* and *value* features to confirm that its vertex is $(1, 2)$ and that it passes through the point $(3, -6)$.

CHECKPOINT Now try Exercise 39.

Study Tip

In Example 4, there are infinitely many different parabolas that have a vertex at $(1, 2)$. Of these, however, the only one that passes through the point $(3, -6)$ is the one given by

$$f(x) = -2(x - 1)^2 + 2.$$

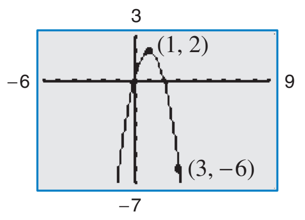


Figure 2.6

Finding Minimum and Maximum Values

Many applications involve finding the maximum or minimum value of a quadratic function. By completing the square of the quadratic function $f(x) = ax^2 + bx + c$, you can rewrite the function in standard form.

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{Standard form}$$

So, the vertex of the graph of f is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, which implies the following.

Minimum and Maximum Values of Quadratic Functions

Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

1. If $a > 0$, then f has a *minimum* at $x = -\frac{b}{2a}$.

The minimum value is $f\left(-\frac{b}{2a}\right)$.

2. If $a < 0$, then f has a *maximum* at $x = -\frac{b}{2a}$.

The maximum value is $f\left(-\frac{b}{2a}\right)$.

Example 5 The Maximum Height of a Projectile

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by the function $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?

Algebraic Solution

For this quadratic function, you have

$$f(x) = ax^2 + bx + c = -0.0032x^2 + x + 3$$

which implies that $a = -0.0032$ and $b = 1$. Because the function has a maximum when $x = -b/(2a)$, you can conclude that the baseball reaches its maximum height when it is x feet from home plate, where x is

$$x = -\frac{b}{2a}$$

$$= -\frac{1}{2(-0.0032)}$$

$$= 156.25 \text{ feet.}$$

At this distance, the maximum height is

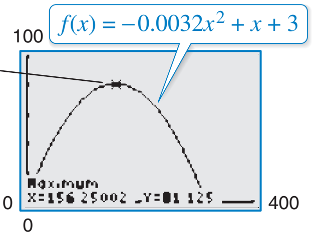
$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3$$

$$= 81.125 \text{ feet.}$$

CHECKPOINT Now try Exercise 65.

Graphical Solution

The maximum height is $y \approx 81.125$ feet at $x \approx 156.25$ feet.



2.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

1. A polynomial function of degree n and leading coefficient a_n is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,\ a_n \neq 0$, where n is a ________ and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are ________ numbers.

2. A ________ function is a second-degree polynomial function, and its graph is called a ________ .

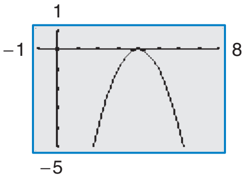
3. Is the quadratic function $f(x) = (x - 2)^2 + 3$ written in standard form? Identify the vertex of the graph of f.

4. Does the graph of the quadratic function $f(x) = -3x^2 + 5x + 2$ have a relative minimum value at its vertex?

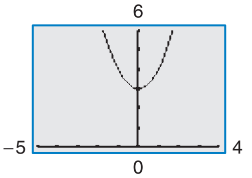
Procedures and Problem Solving

Graphs of Quadratic Functions **In Exercises 5–8, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

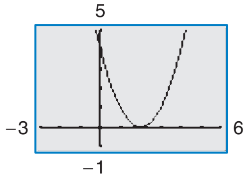
(a)



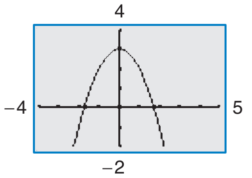
(b)



(c)



(d)



5. $f(x) = (x - 2)^2$

6. $f(x) = 3 - x^2$

7. $f(x) = x^2 + 3$

8. $f(x) = -(x - 4)^2$

Library of Parent Functions **In Exercises 9–16, sketch the graph of the function and describe how the graph is related to the graph of $y = x^2$.**

9. $y = -x^2$

10. $y = x^2 - 1$

11. $y = (x + 3)^2$

12. $y = -(x + 3)^2 - 1$

13. $y = (x + 1)^2$

14. $y = -x^2 + 2$

15. $y = (x - 3)^2$

16. $y = -(x - 3)^2 + 1$

Identifying the Vertex of a Quadratic Function **In Exercises 17–30, describe the graph of the function and identify the vertex. Use a graphing utility to verify your results.**

17. $f(x) = 25 - x^2$

18. $f(x) = x^2 - 7$

19. $f(x) = \frac{1}{2}x^2 - 4$

20. $f(x) = 16 - \frac{1}{4}x^2$

21. $f(x) = (x + 4)^2 - 3$

22. $f(x) = (x - 6)^2 + 3$

23. $h(x) = x^2 - 8x + 16$

24. $g(x) = x^2 + 2x + 1$

25. $f(x) = x^2 - x + \frac{5}{4}$

26. $f(x) = x^2 + 3x + \frac{1}{4}$

27. $f(x) = -x^2 + 2x + 5$

28. $f(x) = -x^2 - 4x + 1$

29. $h(x) = 4x^2 - 4x + 21$

30. $f(x) = 2x^2 - x + 1$

Identifying x-Intercepts of a Quadratic Function **In Exercises 31–36, describe the graph of the quadratic function. Identify the vertex and x-intercept(s). Use a graphing utility to verify your results.**

31. $f(x) = -(x^2 + 2x - 3)$

32. $f(x) = -(x^2 + x - 30)$

33. $g(x) = x^2 + 8x + 11$

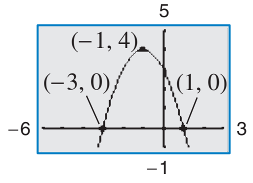
34. $f(x) = x^2 + 10x + 14$

35. $f(x) = -2x^2 + 16x - 31$

36. $f(x) = -4x^2 + 24x - 41$

Writing the Equation of a Parabola in Standard Form **In Exercises 37 and 38, write an equation of the parabola in standard form. Use a graphing utility to graph the equation and verify your result.**

37.



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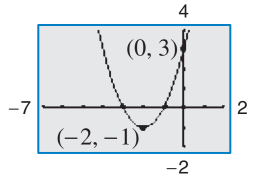
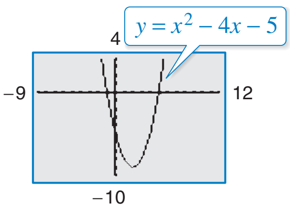


Writing the Equation of a Parabola in Standard Form **In Exercises 39–44, write the standard form of the quadratic function that has the indicated vertex and whose graph passes through the given point. Use a graphing utility to verify your result.**

39. Vertex: $(-2, 5)$;	Point: $(0, 9)$
40. Vertex: $(4, 1)$;	Point: $(6, -7)$
41. Vertex: $(1, -2)$;	Point: $(-1, 14)$
42. Vertex: $(-4, -1)$;	Point: $(-2, 4)$
43. Vertex: $\left(\frac{1}{2}, 1\right)$;	Point: $\left(-2, -\frac{21}{5}\right)$
44. Vertex: $\left(-\frac{1}{4}, -1\right)$;	Point: $\left(0, -\frac{17}{16}\right)$

Using a Graph to Identify *x*-Intercepts **In Exercises 45–48, determine the *x*-intercept(s) of the graph visually. Then find the *x*-intercept(s) algebraically to verify your answer.**

45.



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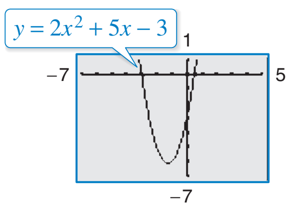

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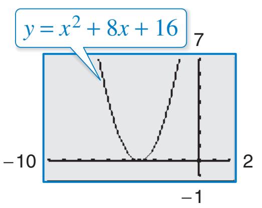

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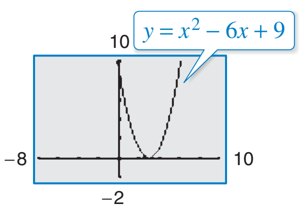

Graphing to Identify *x*-Intercepts **In Exercises 49–54, use a graphing utility to graph the quadratic function and find the *x*-intercepts of the graph. Then find the *x*-intercepts algebraically to verify your answer.**

49. $y = x^2 - 4x$	**50.** $y = -2x^2 + 10x$
51. $y = 2x^2 - 7x - 30$	**52.** $y = 4x^2 + 25x - 21$
53. $y = -\frac{1}{2}(x^2 - 6x - 7)$	**54.** $y = \frac{7}{10}(x^2 + 12x - 45)$

Using the *x*-Intercepts to Write Equations **In Exercises 55–58, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given *x*-intercepts. (There are many correct answers.)**

55. $(-1, 0), (3, 0)$	**56.** $(0, 0), (10, 0)$
57. $(-3, 0), \left(-\frac{1}{2}, 0\right)$	**58.** $\left(-\frac{5}{2}, 0\right), (2, 0)$

Maximizing a Product of Two Numbers **In Exercises 59–62, find the two positive real numbers with the given sum whose product is a maximum.**

59. The sum is 110.	**60.** The sum is 66.

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61. The sum of the first and twice the second is 24.

62. The sum of the first and three times the second is 42.

63. ***Why you should learn it*** *(p. 90)* An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter single-lane running track.

(a) Draw a diagram that illustrates the problem. Let x and y represent the length and width of the rectangular region, respectively.

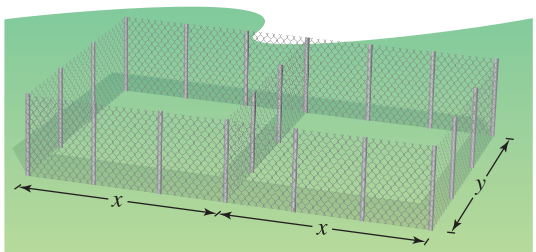
(b) Determine the radius of the semicircular ends of the track. Determine the distance, in terms of y, around the inside edge of the two semicircular parts of the track.

(c) Use the result of part (b) to write an equation, in terms of x and y, for the distance traveled in one lap around the track. Solve for y.

(d) Use the result of part (c) to write the area A of the rectangular region as a function of x.

(e) Use a graphing utility to graph the area function from part (d). Use the graph to approximate the dimensions that will produce a rectangle of maximum area.

64. **Algebraic-Graphical-Numerical** A child care center has 200 feet of fencing to enclose two adjacent rectangular safe play areas (see figure). Use the following methods to determine the dimensions that will produce a maximum enclosed area.



(a) Write the total area A of the play areas as a function of x.

(b) Use the *table* feature of a graphing utility to create a table showing possible values of x and the corresponding total area A of the play areas. Use the table to estimate the dimensions that will produce the maximum enclosed area.

(c) Use the graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum enclosed area.

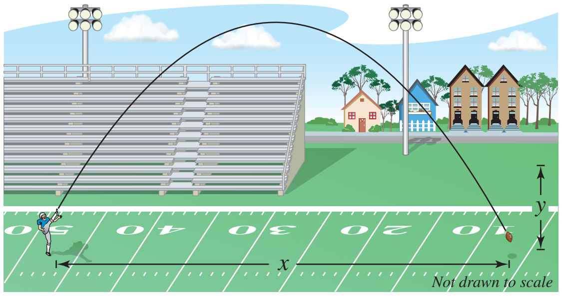
(d) Write the area function in standard form to find algebraically the dimensions that will produce the maximum enclosed area.

(e) Compare your results from parts (b), (c), and (d).

✓ **65. Height of a Projectile** The height y (in feet) of a punted football is approximated by

$$y = -\frac{16}{2025}x^2 + \frac{9}{5}x + \frac{3}{2}$$

where x is the horizontal distance (in feet) from where the football is punted.



(a) Use a graphing utility to graph the path of the football.

(b) How high is the football when it is punted? (*Hint:* Find y when $x = 0$.)

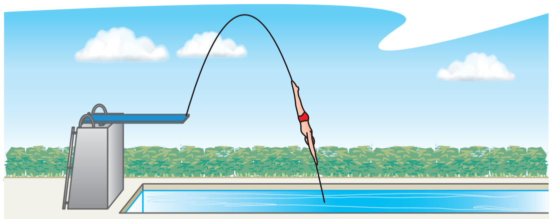
(c) What is the maximum height of the football?

(d) How far from the punter does the football strike the ground?

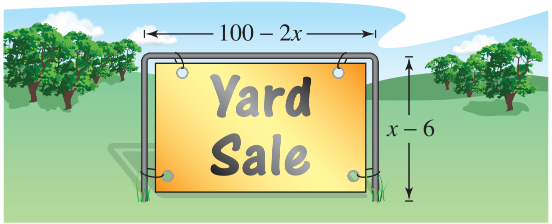
66. Physics The path of a diver is approximated by

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where y is the height (in feet) and x is the horizontal distance (in feet) from the end of the diving board (see figure). What is the maximum height of the diver? Verify your answer using a graphing utility.



67. Geometry To make a sign holder, you bend a 100-inch long steel wire x inches from each end to form two right angles. To use the sign holder, you insert each end 6 inches into the ground.



(a) Write a function for the rectangular area A enclosed by the sign holder in terms of x.

(b) Use the *table* feature of a graphing utility to determine the value of x that maximizes the rectangular area enclosed by the sign holder.

68. Aerodynamic Engineering The number of horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(s) = 0.002s^2 + 0.05s - 0.029, \quad 0 \le s \le 100$$

where s is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Graphically estimate the maximum speed of the car given that the power required to overcome wind drag is not to exceed 10 horsepower. Verify your result algebraically.

69. Economics The monthly revenue R (in thousands of dollars) from the sales of a digital picture frame is approximated by $R(p) = -10p^2 + 1580p$, where p is the price per unit (in dollars).

(a) Find the monthly revenues for unit prices of \$50, \$70, and \$90.

(b) Find the unit price that will yield a maximum monthly revenue.

(c) What is the maximum monthly revenue?

(d) Explain your results.

70. Economics The weekly revenue R (in dollars) earned by a computer repair service is given by

$$R(p) = -12p^2 + 372p$$

where p is the price charged per service hour (in dollars).

(a) Find the weekly revenues for prices per service hour of \$12, \$16, and \$20.

(b) Find the price that will yield a maximum weekly revenue.

(c) What is the maximum weekly revenue?

(d) Explain your results.

71. Public Health From 1955 through 2000, the annual per capita consumption C of cigarettes by Americans (age 18 and older) can be modeled by

$$C(t) = -2.10t^2 + 70.9t + 3557, \quad 5 \le t \le 50$$

where t is the year, with $t = 5$ corresponding to 1955. (Source: U.S. Department of Agriculture)

(a) Use a graphing utility to graph the model.

(b) Use the graph of the model to approximate the year when the maximum annual consumption of cigarettes occurred. Approximate the maximum average annual consumption. Beginning in 1966, all cigarette packages were required by law to carry a health warning. Do you think the warning had any effect? Explain.

(c) In 2000, the U.S. population (age 18 and older) was 209,117,000. Of those, about 48,306,000 were smokers. What was the average annual cigarette consumption *per smoker* in 2000? What was the average daily cigarette consumption *per smoker*?

72. Demography The population P of Germany (in thousands) from 1999 through 2009 can be modeled by

$$P(t) = -8.87t^2 + 271.4t + 80{,}362, \quad 9 \le t \le 19$$

where t is the year, with $t = 9$ corresponding to 1999. (Source: U.S. Census Bureau)

(a) According to the model, in what year did Germany have its greatest population? What was the population?

(b) According to the model, what will Germany's population be in the year 2100? Is this result reasonable? Explain.

Conclusions

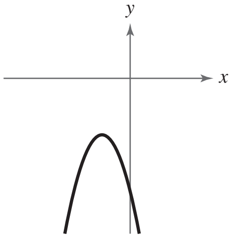
True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. The function $f(x) = -12x^2 - 1$ has no x-intercepts.

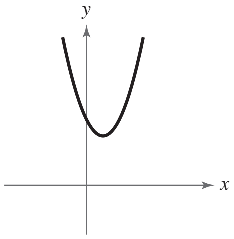
74. The graphs of $f(x) = -4x^2 - 10x + 7$ and $g(x) = 12x^2 + 30x + 1$ have the same axis of symmetry.

Library of Parent Functions **In Exercises 75 and 76, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)**

75. (a) $f(x) = -(x - 4)^2 + 2$
(b) $f(x) = -(x + 2)^2 + 4$
(c) $f(x) = -(x + 2)^2 - 4$
(d) $f(x) = -x^2 - 4x - 8$
(e) $f(x) = -(x - 2)^2 - 4$
(f) $f(x) = -x^2 + 4x - 8$



76. (a) $f(x) = (x - 1)^2 + 3$
(b) $f(x) = (x + 1)^2 + 3$
(c) $f(x) = (x - 3)^2 + 1$
(d) $f(x) = x^2 + 2x + 4$
(e) $f(x) = (x + 3)^2 + 1$
(f) $f(x) = x^2 + 6x + 10$



Describing Parabolas **In Exercises 77–80, let z represent a positive real number. Describe how the family of parabolas represented by the given function compares with the graph of $g(x) = x^2$.**

77. $f(x) = (x - z)^2$

78. $f(x) = x^2 - z$

79. $f(x) = z(x - 3)^2$

80. $f(x) = zx^2 + 4$

Think About It **In Exercises 81–84, find the value of b such that the function has the given maximum or minimum value.**

81. $f(x) = -x^2 + bx - 75$; Maximum value: 25

82. $f(x) = -x^2 + bx - 16$; Maximum value: 48

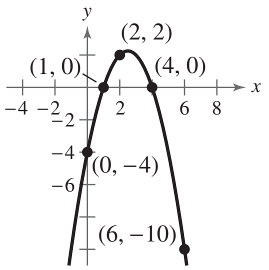
83. $f(x) = x^2 + bx + 26$; Minimum value: 10

84. $f(x) = x^2 + bx - 25$; Minimum value: -50

85. Proof Let x and y be two positive real numbers whose sum is S. Show that the maximum product of x and y occurs when x and y are both equal to $S/2$.

86. Proof Assume that the function given by $f(x) = ax^2 + bx + c$, $a \ne 0$, has two real zeros. Show that the x-coordinate of the vertex of the graph is the average of the zeros of f. (*Hint:* Use the Quadratic Formula.)

87. Writing The parabola in the figure below has an equation of the form $y = ax^2 + bx - 4$. Find the equation of this parabola in two different ways, by hand and with technology (graphing utility or computer software). Write a paragraph describing the methods you used and comparing the results of the two methods.



88. CAPSTONE The annual profit P (in dollars) of a company is modeled by a function of the form $P = at^2 + bt + c$, where t represents the year. Discuss which of the following models the company might prefer.

(a) a is positive and $t \ge -b/(2a)$.

(b) a is positive and $t \le -b/(2a)$.

(c) a is negative and $t \ge -b/(2a)$.

(d) a is negative and $t \le -b/(2a)$.

Cumulative Mixed Review

Finding Points of Intersection **In Exercises 89–92, determine algebraically any point(s) of intersection of the graphs of the equations. Verify your results using the *intersect* feature of a graphing utility.**

89. $x + y = 8$
$-\frac{2}{3}x + y = 6$

90. $y = 3x - 10$
$y = \frac{1}{4}x + 1$

91. $y = 9 - x^2$
$y = x + 3$

92. $y = x^3 + 2x - 1$
$y = -2x + 15$

93. *Make a Decision* To work an extended application analyzing the height of a basketball after it has been dropped, visit this textbook's *Companion Website*.

2.2 Polynomial Functions of Higher Degree

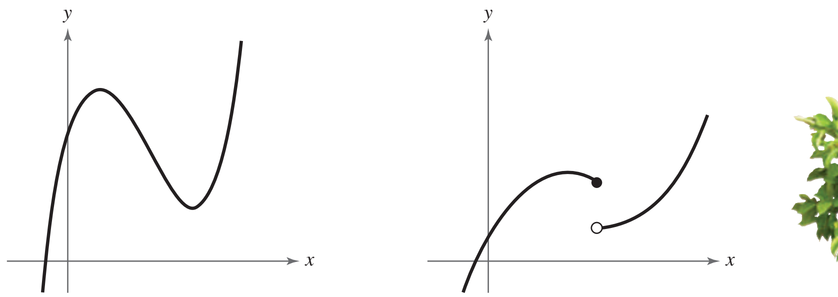
Graphs of Polynomial Functions

At this point, you should be able to sketch accurate graphs of polynomial functions of degrees 0, 1, and 2.

Function	*Graph*
$f(x) = a$	Horizontal line
$f(x) = ax + b$	Line of slope a
$f(x) = ax^2 + bx + c$	Parabola

The graphs of polynomial functions of degree greater than 2 are more difficult to sketch by hand. However, in this section you will learn how to recognize some of the basic features of the graphs of polynomial functions. Using these features along with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

The graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.7. Informally, you can say that a function is continuous when its graph can be drawn with a pencil without lifting the pencil from the paper.

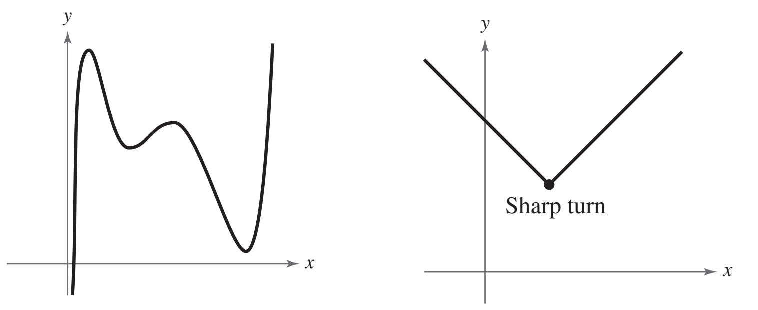


(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 2.7

Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 2.8(a). It cannot have a sharp turn such as the one shown in Figure 2.8(b).



(a) Polynomial functions have graphs with smooth, rounded turns.

(b) Functions with graphs that have sharp turns are not polynomial functions.

Figure 2.8

What you should learn

- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions.
- Find and use zeros of polynomial functions as sketching aids.
- Use the Intermediate Value Theorem to help locate zeros of polynomial functions.

Why you should learn it

You can use polynomial functions to model various aspects of nature, such as the growth of a red oak tree, as shown in Exercise 112 on page 111.

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The graphs of polynomial functions of degree 1 are lines, and those of functions of degree 2 are parabolas. The graphs of all polynomial functions are smooth and continuous. A polynomial function of degree n has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

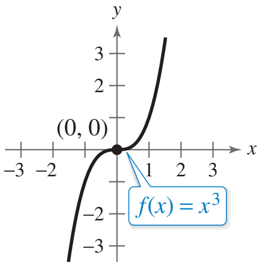
where n is a positive integer and $a_n \neq 0$.

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero. The greater the value of n, the flatter the graph near the origin. When n is even, the graph is similar to the graph of $f(x) = x^2$ and touches the x-axis at the x-intercept. When n is odd, the graph is similar to the graph of $f(x) = x^3$ and crosses the x-axis at the x-intercept. Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions.**

Library of Parent Functions: Cubic Function

The basic characteristics of the *parent cubic function* $f(x) = x^3$ are summarized below and on the inside cover of this text.

Graph of $f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing on $(-\infty, \infty)$
Odd function
Origin symmetry



Explore the Concept

Use a graphing utility to graph $y = x^n$ for $n = 2$, 4, and 8. (Use the viewing window $-1.5 \leq x \leq 1.5$ and $-1 \leq y \leq 6$.) Compare the graphs. In the interval $(-1, 1)$, which graph is on the bottom? Outside the interval $(-1, 1)$, which graph is on the bottom?

Use a graphing utility to graph $y = x^n$ for $n = 3$, 5, and 7. (Use the viewing window $-1.5 \leq x \leq 1.5$ and $-4 \leq y \leq 4$.) Compare the graphs. In the intervals $(-\infty, -1)$ and $(0, 1)$, which graph is on the bottom? In the intervals $(-1, 0)$ and $(1, \infty)$, which graph is on the bottom?

Example 1 Library of Parent Functions: $f(x) = x^3$

Sketch the graphs of (a) $g(x) = -x^3$, (b) $h(x) = x^3 + 1$, and (c) $k(x) = (x - 1)^3$.

Solution

a. With respect to the graph of $f(x) = x^3$, the graph of g is obtained by a *reflection* in the x-axis, as shown in Figure 2.9.

b. With respect to the graph of $f(x) = x^3$, the graph of h is obtained by a vertical shift one unit *upward*, as shown in Figure 2.10.

c. With respect to the graph of $f(x) = x^3$, the graph of k is obtained by a horizontal shift one unit *to the right*, as shown in Figure 2.11.

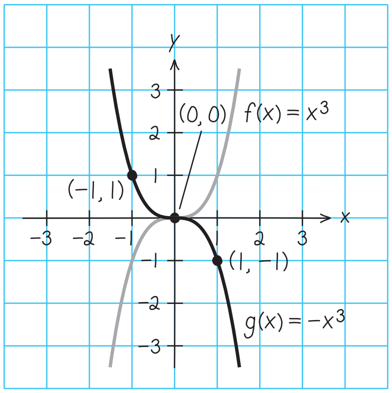


Figure 2.9

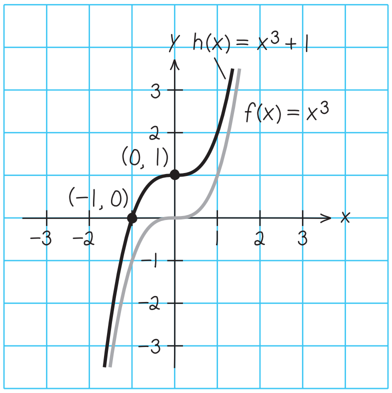


Figure 2.10

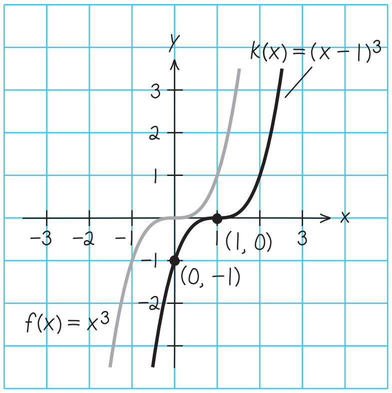


Figure 2.11

CHECKPOINT Now try Exercise 17.

The Leading Coefficient Test

In Example 1, note that all three graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial eventually rises or falls can be determined by the polynomial function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

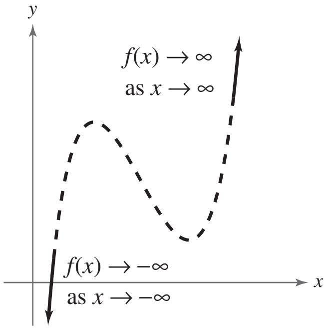
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function

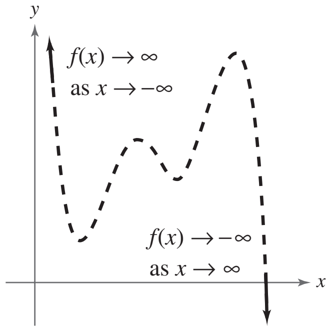
$$f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0, \quad a_n \neq 0$$

eventually rises or falls in the following manner.

1. When n is odd:

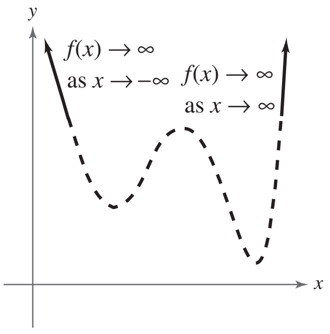


If the leading coefficient is positive ($a_n > 0$), then the graph falls to the left and rises to the right.

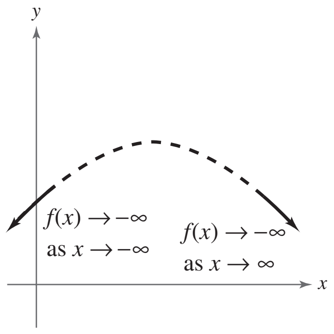


If the leading coefficient is negative ($a_n < 0$), then the graph rises to the left and falls to the right.

2. When n is even:



If the leading coefficient is positive ($a_n > 0$), then the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), then the graph falls to the left and right.

Note that the dashed portions of the graphs indicate that the test determines only the right-hand and left-hand behavior of the graph.

As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

Explore the Concept

For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree and sign of the leading coefficient of the function, and the right- and left-hand behavior of the graph of the function.

a. $y = x^3 - 2x^2 - x + 1$

b. $y = 2x^5 + 2x^2 - 5x + 1$

c. $y = -2x^5 - x^2 + 5x + 3$

d. $y = -x^3 + 5x - 2$

e. $y = 2x^2 + 3x - 4$

f. $y = x^4 - 3x^2 + 2x - 1$

g. $y = -x^2 + 3x + 2$

h. $y = -x^6 - x^2 - 5x + 4$

Study Tip

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

Example 2 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of

$$f(x) = -x^3 + 4x.$$

Solution

Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 2.12.

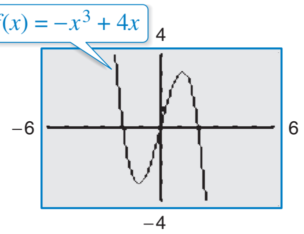


Figure 2.12

CHECKPOINT Now try Exercise 29.

Example 3 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of each polynomial function.

a. $f(x) = x^4 - 5x^2 + 4$

b. $f(x) = x^5 - x$

Solution

a. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 2.13.

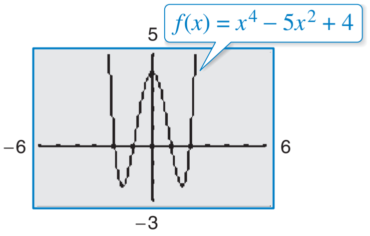


Figure 2.13

b. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 2.14.

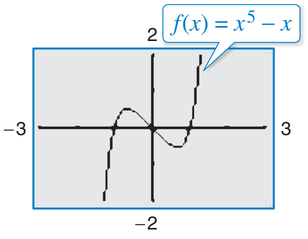


Figure 2.14

CHECKPOINT Now try Exercise 31.

In Examples 2 and 3, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Explore the Concept

For each of the graphs in Examples 2 and 3, count the number of zeros of the polynomial function and the number of relative extrema, and compare these numbers with the degree of the polynomial. What do you observe?

Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The function f has at most n real zeros. (You will study this result in detail in Section 2.5 on the Fundamental Theorem of Algebra.)
2. The graph of f has at most $n - 1$ relative **extrema** (relative minima or maxima).

Recall that a zero of a function f is a number x for which $f(x) = 0$. Finding the zeros of polynomial functions is one of the most important problems in algebra. You have already seen that there is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros. In other cases, you can use information about the zeros of a function to find a good viewing window.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, then the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

Finding zeros of polynomial functions is closely related to factoring and finding x-intercepts, as demonstrated in Examples 4, 5, and 6.

Example 4 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Algebraic Solution

$f(x) = x^3 - x^2 - 2x$ — Write original function.

$0 = x^3 - x^2 - 2x$ — Substitute 0 for $f(x)$.

$0 = x(x^2 - x - 2)$ — Remove common monomial factor.

$0 = x(x - 2)(x + 1)$ — Factor completely.

So, the real zeros are

$$x = 0, \quad x = 2, \quad \text{and} \quad x = -1$$

and the corresponding x-intercepts are

$$(0, 0), \quad (2, 0), \quad \text{and} \quad (-1, 0).$$

Check

$(0)^3 - (0)^2 - 2(0) = 0$ — $x = 0$ is a zero. ✓

$(2)^3 - (2)^2 - 2(2) = 0$ — $x = 2$ is a zero. ✓

$(-1)^3 - (-1)^2 - 2(-1) = 0$ — $x = -1$ is a zero. ✓

CHECKPOINT Now try Exercise 37.

Graphical Solution

The graph of f has the x-intercepts

$$(0, 0), \quad (2, 0), \quad \text{and} \quad (-1, 0)$$

as shown in Figure 2.15. So, the real zeros of f are

$$x = 0, \quad x = 2, \quad \text{and} \quad x = -1.$$

Use the *zero* or *root* feature of a graphing utility to verify these zeros.

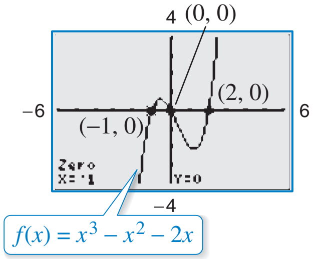


Figure 2.15

Example 5 Analyzing a Polynomial Function

Find all real zeros and relative extrema of $f(x) = -2x^4 + 2x^2$.

Solution

$$0 = -2x^4 + 2x^2 \qquad \text{Substitute 0 for } f(x).$$

$$0 = -2x^2(x^2 - 1) \qquad \text{Remove common monomial factor.}$$

$$0 = -2x^2(x - 1)(x + 1) \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 2.16. Using the *minimum* and *maximum* features of a graphing utility, you can approximate the three relative extrema to be $(-0.71, 0.5)$, $(0, 0)$, and $(0.71, 0.5)$.

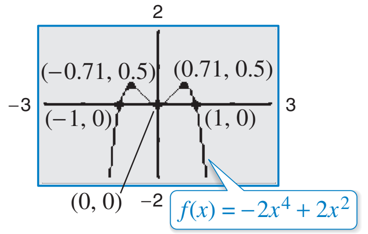


Figure 2.16

CHECKPOINT Now try Exercise 59.

> **Repeated Zeros**
>
> For a polynomial function, a factor of $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.
>
> **1.** If k is odd, then the graph *crosses* the x-axis at $x = a$.
>
> **2.** If k is even, then the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

Study Tip

In Example 5, note that because k is even, the factor $-2x^2$ yields the repeated zero $x = 0$. The graph touches (but does not cross) the x-axis at $x = 0$, as shown in Figure 2.16.

Example 6 Analyzing a Polynomial Function

Find all real zeros of $f(x) = x^5 - 3x^3 - x^2 - 4x - 1$.

Solution

From Figure 2.17, you can see that there are three zeros. Using the *zero* feature of a graphing utility, you can determine that the zeros are approximately $x \approx -1.86$, $x \approx -0.25$, and $x \approx 2.11$. It should be noted that this fifth-degree polynomial factors as

$$f(x) = x^5 - 3x^3 - x^2 - 4x - 1 = (x^2 + 1)(x^3 - 4x - 1).$$

The three zeros obtained above are the zeros of the cubic factor $x^3 - 4x - 1$. The quadratic factor $x^2 + 1$ has no real zeros, but does have two *complex* zeros. You will learn more about complex zeros in Section 2.5.

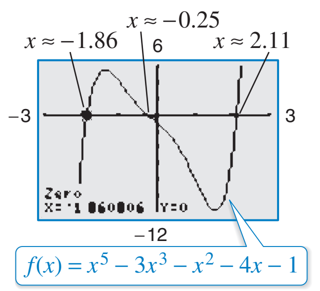


Figure 2.17

CHECKPOINT Now try Exercise 61.

Example 7 Finding a Polynomial Function with Given Zeros

Find a polynomial function with zeros $-\frac{1}{2}$, 3, and 3. (There are many correct solutions.)

Solution

Note that the zero $x = -\frac{1}{2}$ corresponds to either $\left(x + \frac{1}{2}\right)$ or $(2x + 1)$. To avoid fractions, choose the second factor and write

$$\begin{aligned} f(x) &= (2x + 1)(x - 3)^2 \\ &= (2x + 1)(x^2 - 6x + 9) \\ &= 2x^3 - 11x^2 + 12x + 9. \end{aligned}$$

CHECKPOINT Now try Exercise 67.

Note in Example 7 that there are many polynomial functions with the indicated zeros. In fact, multiplying the function by any real number does not change the zeros of the function. For instance, multiply the function from Example 7 by $\frac{1}{2}$ to obtain

$$f(x) = x^3 - \tfrac{11}{2}x^2 + 6x + \tfrac{9}{2}.$$

Then find the zeros of the function. You will obtain the zeros $-\frac{1}{2}$, 3, and 3, as given in Example 7.

Example 8 Sketching the Graph of a Polynomial Function

Sketch the graph of

$$f(x) = 3x^4 - 4x^3$$

by hand.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 2.18).
2. *Find the Real Zeros of the Polynomial.* By factoring

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4)$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 3) and $x = \frac{4}{3}$ (of odd multiplicity 1). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 2.18.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Be sure to choose points between the zeros and to the left and right of the zeros. Then plot the points (see Figure 2.19).

x	-1	0.5	1	1.5
$f(x)$	7	-0.31	-1	1.69

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.19. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$. When you are unsure of the shape of a portion of the graph, plot some additional points.

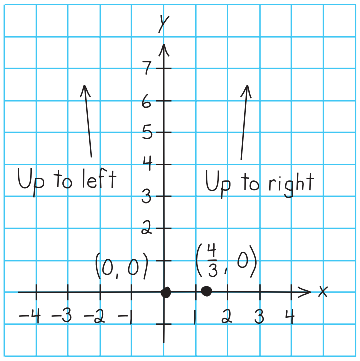


Figure 2.18

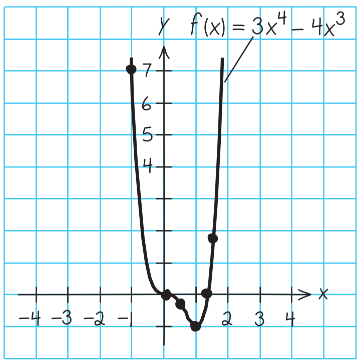
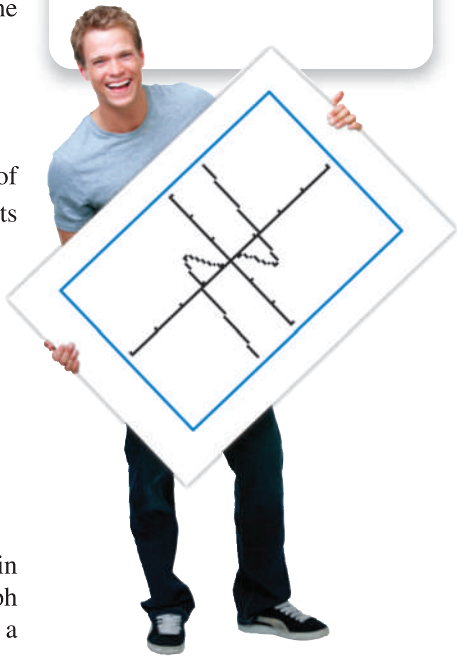


Figure 2.19

CHECKPOINT Now try Exercise 87.

Technology Tip

Because it is easy to make mistakes when entering functions into a graphing utility, it is important to understand the basic shapes of graphs and to be able to graph simple polynomials *by hand*. For instance, suppose you had entered the function in Example 8 as $y = 3x^5 - 4x^3$. From the graph, what mathematical principles would alert you to the fact that you had made a mistake?



Explore the Concept

Partner Activity Multiply three, four, or five distinct linear factors to obtain the equation of a polynomial function of degree 3, 4, or 5. Exchange equations with your partner and sketch, *by hand*, the graph of the equation that your partner wrote. When you are finished, use a graphing utility to check each other's work.

Yuri Arcurs 2010/used under license from Shutterstock.com

Example 9 Sketching the Graph of a Polynomial Function

Sketch the graph of

$$f(x) = -2x^3 + 6x^2 - \frac{9}{2}x.$$

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 2.20).

2. *Find the Real Zeros of the Polynomial.* By factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 1) and $x = \frac{3}{2}$ (of even multiplicity 2). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 2.20.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points (see Figure 2.21).

x	-0.5	0.5	1	2
$f(x)$	4	-1	-0.5	-1

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.21. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ and touches (but does not cross) the x-axis at $\left(\frac{3}{2}, 0\right)$.

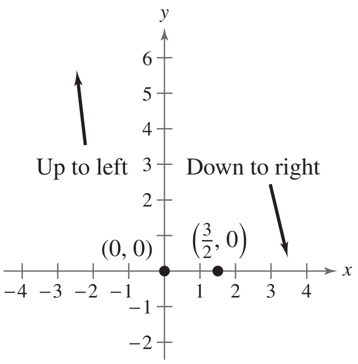


Figure 2.20

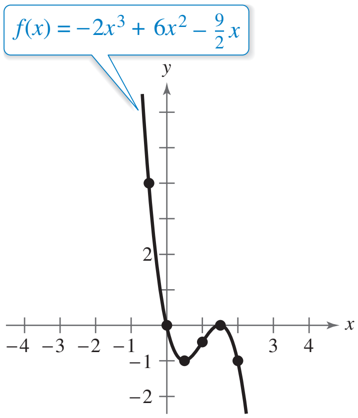


Figure 2.21

CHECKPOINT Now try Exercise 89.

Study Tip

Observe in Example 9 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if a zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side of the zero to the other side. If a zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side of the zero to the other side. The following table helps to illustrate this result.

x	-0.5	0	0.5
$f(x)$	4	0	-1
Sign	$+$		$-$

x	1	$\frac{3}{2}$	2
$f(x)$	-0.5	0	-1
Sign	$-$		$-$

This sign analysis may be helpful in graphing polynomial functions.

Technology Tip

Remember that when using a graphing utility to verify your graphs, you may need to adjust your viewing window in order to see all the features of the graph.

The Intermediate Value Theorem

The **Intermediate Value Theorem** concerns the existence of real zeros of polynomial functions. The theorem states that if

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 2.22.)

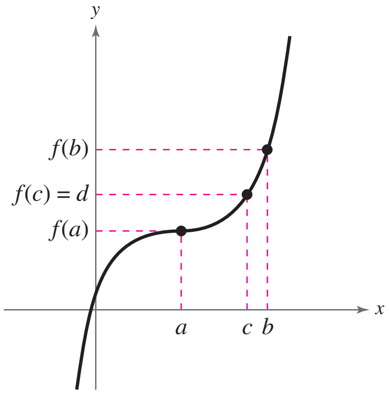


Figure 2.22

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

This theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, then you can conclude that the function has at least one real zero between these two values. For example, the function $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1.

Example 10 Approximating the Zeros of a Function

Find three intervals of length 1 in which the polynomial

$$f(x) = 12x^3 - 32x^2 + 3x + 5$$

is guaranteed to have a zero.

Graphical Solution

From Figure 2.23, you can see that the graph of f crosses the x-axis three times—between -1 and 0, between 0 and 1, and between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

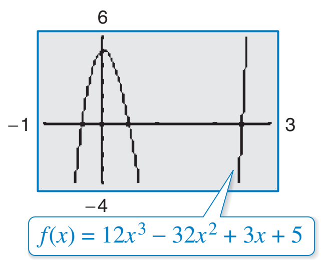


Figure 2.23

CHECKPOINT Now try Exercise 95.

Numerical Solution

From the table in Figure 2.24, you can see that $f(-1)$ and $f(0)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between -1 and 0. Similarly, $f(0)$ and $f(1)$ differ in sign, so the function has a zero between 0 and 1. Likewise, $f(2)$ and $f(3)$ differ in sign, so the function has a zero between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

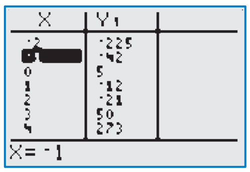


Figure 2.24

2.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. The graph of a polynomial function is ________ , so it has no breaks, holes, or gaps.

2. A polynomial function of degree n has at most ________ real zeros and at most ________ relative extrema.

3. If $x = a$ is a zero of a polynomial function f, then the following statements are true.

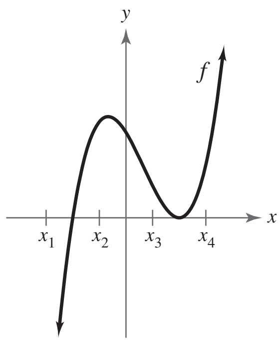
(a) $x = a$ is a ________ of the polynomial equation $f(x) = 0$.

(b) ________ is a factor of the polynomial $f(x)$.

(c) The point ________ is an x-intercept of the graph of f.

4. If a zero of a polynomial function f is of even multiplicity, then the graph of f ________ the x-axis, and if the zero is of odd multiplicity, then the graph of f ________ the x-axis.

For Exercises 5–8, the graph shows the right-hand and left-hand behavior of a polynomial function f.



5. Can f be a fourth-degree polynomial function?

6. Can the leading coefficient of f be negative?

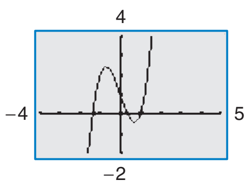
7. The graph shows that $f(x_1) < 0$. What other information shown in the graph allows you to apply the Intermediate Value Theorem to guarantee that f has a zero in the interval $[x_1, x_2]$?

8. Is the repeated zero of f in the interval $[x_3, x_4]$ of even or odd multiplicity?

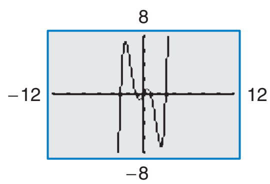
Procedures and Problem Solving

Identifying Graphs of Polynomial Functions **In Exercises 9–16, match the polynomial function with its graph. [The graphs are labeled (a) through (h).]**

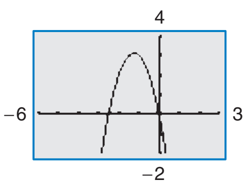
(a)



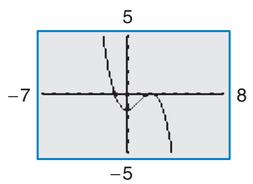
(b)



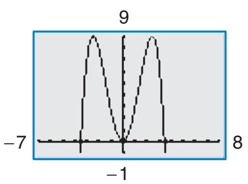
(c)



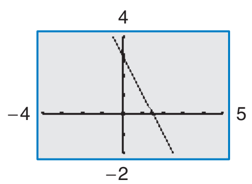
(d)



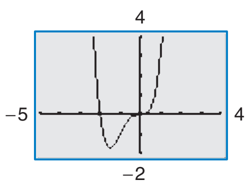
(e)



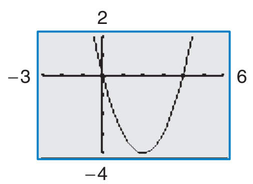
(f)



(g)



(h)



9. $f(x) = -2x + 3$

10. $f(x) = x^2 - 4x$

11. $f(x) = -2x^2 - 5x$

12. $f(x) = 2x^3 - 3x + 1$

13. $f(x) = -\frac{1}{4}x^4 + 3x^2$

14. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$

15. $f(x) = x^4 + 2x^3$

16. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

Library of Parent Functions **In Exercises 17–22, sketch the graph of $y = x^3$ and the graph of the function f. Describe the transformation from y to f.**

✓ **17.** $f(x) = (x - 2)^3$

18. $f(x) = x^3 - 2$

19. $f(x) = -x^3 + 1$

20. $f(x) = (x - 2)^3 - 2$

21. $f(x) = -(x - 2)^3$

22. $f(x) = -x^3 + 3$

Comparing End Behavior **In Exercises 23–28, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough to see the right-hand and left-hand behavior of each graph. Do the graphs of f and g have the same right-hand and left-hand behavior? Explain why or why not.**

23. $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$

24. $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$

25. $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$

26. $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

27. $f(x) = -2x^3 + 4x^2 - 1, \quad g(x) = 2x^3$

28. $f(x) = -(x^4 - 6x^2 - x + 10), \quad g(x) = x^4$

Applying the Leading Coefficient Test **In Exercises 29–36, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function. Use a graphing utility to verify your result.**

29. $f(x) = 2x^4 - 3x + 1$ **30.** $h(x) = 1 - x^6$

31. $g(x) = 5 - \frac{7}{2}x - 3x^2$ **32.** $f(x) = \frac{1}{3}x^3 + 5x$

33. $f(x) = \dfrac{6x^5 - 2x^4 + 4x^2 - 5x}{3}$

34. $f(x) = \dfrac{3x^7 - 2x^5 + 5x^3 + 6x^2}{4}$

35. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

36. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

Finding Zeros of a Polynomial Function **In Exercises 37–48, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those from part (a).**

37. $f(x) = 3x^2 - 12x + 3$ **38.** $g(x) = 5x^2 - 10x - 5$

39. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$ **40.** $y = \frac{1}{4}x^3(x^2 - 9)$

41. $f(x) = x^5 + x^3 - 6x$ **42.** $g(t) = t^5 - 6t^3 + 9t$

43. $f(x) = 2x^4 - 2x^2 - 40$

44. $f(x) = 5x^4 + 15x^2 + 10$

45. $f(x) = x^3 - 4x^2 - 25x + 100$

46. $y = 4x^3 + 4x^2 - 7x + 2$

47. $y = 4x^3 - 20x^2 + 25x$

48. $y = x^5 - 5x^3 + 4x$

Finding Zeros and Their Multiplicities **In Exercises 49–58, find all the real zeros of the polynomial function. Determine the multiplicity of each zero. Use a graphing utility to verify your result.**

49. $f(x) = x^2 - 25$ **50.** $f(x) = 49 - x^2$

51. $h(t) = t^2 - 6t + 9$ **52.** $f(x) = x^2 + 10x + 25$

53. $f(x) = x^2 + x - 2$ **54.** $f(x) = 2x^2 - 14x + 24$

55. $f(t) = t^3 - 4t^2 + 4t$ **56.** $f(x) = x^4 - x^3 - 20x^2$

57. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$ **58.** $f(x) = \frac{5}{3}x^2 + \frac{8}{3}x - \frac{4}{3}$

Analyzing a Polynomial Function **In Exercises 59–64, use a graphing utility to graph the function and approximate (accurate to three decimal places) any real zeros and relative extrema.**

59. $f(x) = 2x^4 - 6x^2 + 1$

60. $f(x) = -\frac{3}{8}x^4 - x^3 + 2x^2 + 5$

61. $f(x) = x^5 + 3x^3 - x + 6$

62. $f(x) = -3x^3 - 4x^2 + x - 3$

63. $f(x) = -2x^4 + 5x^2 - x - 1$

64. $f(x) = 3x^5 - 2x^2 - x + 1$

Finding a Polynomial Function with Given Zeros **In Exercises 65–74, find a polynomial function that has the given zeros. (There are many correct answers.)**

65. $0, 4$ **66.** $-7, 2$

67. $0, -2, -3$ **68.** $0, 2, 5$

69. $4, -3, 3, 0$ **70.** $-2, -1, 0, 1, 2$

71. $1 + \sqrt{3}, 1 - \sqrt{3}$ **72.** $6 + \sqrt{3}, 6 - \sqrt{3}$

73. $2, 4 + \sqrt{5}, 4 - \sqrt{5}$ **74.** $4, 2 + \sqrt{7}, 2 - \sqrt{7}$

Finding a Polynomial Function with Given Zeros **In Exercises 75–80, find a polynomial function with the given zeros, multiplicities, and degree. (There are many correct answers.)**

75. Zero: -2, multiplicity: 2
Zero: -1, multiplicity: 1
Degree: 3

76. Zero: 3, multiplicity: 1
Zero: 2, multiplicity: 3
Degree: 4

77. Zero: -4, multiplicity: 2
Zero: 3, multiplicity: 2
Degree: 4

78. Zero: 5, multiplicity: 3
Zero: 0, multiplicity: 2
Degree: 5

79. Zero: -1, multiplicity: 2
Zero: -2, multiplicity: 1
Degree: 3
Rises to the left,
Falls to the right

80. Zero: 1, multiplicity: 2
Zero: 4, multiplicity: 2
Degree: 4
Falls to the left,
Falls to the right

Sketching a Polynomial with Given Conditions **In Exercises 81–84, sketch the graph of a polynomial function that satisfies the given conditions. If not possible, explain your reasoning. (There are many correct answers.)**

81. Third-degree polynomial with two real zeros and a negative leading coefficient

82. Fourth-degree polynomial with three real zeros and a positive leading coefficient

83. Fifth-degree polynomial with three real zeros and a positive leading coefficient

84. Fourth-degree polynomial with two real zeros and a negative leading coefficient

Sketching the Graph of a Polynomial Function **In Exercises 85–94, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.**

85. $f(x) = x^3 - 9x$ **86.** $g(x) = x^4 - 4x^2$

87. $f(x) = x^3 - 3x^2$ **88.** $f(x) = 3x^3 - 24x^2$

89. $f(x) = -x^4 + 9x^2 - 20$ **90.** $f(x) = -x^6 + 7x^3 + 8$

91. $f(x) = x^3 + 3x^2 - 9x - 27$

92. $h(x) = x^5 - 4x^3 + 8x^2 - 32$

93. $g(t) = -\frac{1}{4}t^4 + 2t^2 - 4$

94. $g(x) = \frac{1}{10}(x^4 - 4x^3 - 2x^2 + 12x + 9)$

Approximating the Zeros of a Function **In Exercises 95–100, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero, and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your answers in part (a) by using the *table* feature of the graphing utility.**

95. $f(x) = x^3 - 3x^2 + 3$ **96.** $f(x) = -2x^3 - 6x^2 + 3$

97. $g(x) = 3x^4 + 4x^3 - 3$ **98.** $h(x) = x^4 - 10x^2 + 2$

99. $f(x) = x^4 - 3x^3 - 4x - 3$

100. $f(x) = x^3 - 4x^2 - 2x + 10$

Identifying Symmetry and x-Intercepts **In Exercises 101–108, use a graphing utility to graph the function. Identify any symmetry with respect to the x-axis, y-axis, or origin. Determine the number of x-intercepts of the graph.**

101. $f(x) = x^2(x + 6)$ **102.** $h(x) = x^3(x - 4)^2$

103. $g(t) = -\frac{1}{2}(t - 4)^2(t + 4)^2$

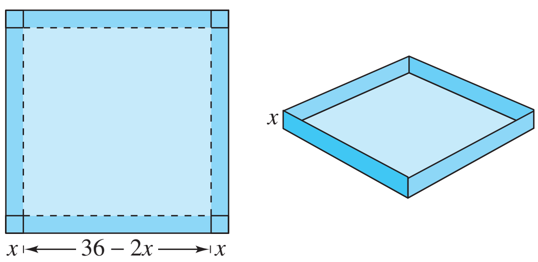
104. $g(x) = \frac{1}{8}(x + 1)^2(x - 3)^3$

105. $f(x) = x^3 - 4x$ **106.** $f(x) = x^4 - 2x^2$

107. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

108. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

109. **Geometry** An open box is to be made from a square piece of material 36 centimeters on a side by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).



(a) Verify that the volume of the box is given by the function $V(x) = x(36 - 2x)^2$.

(b) Determine the domain of the function V.

(c) Use the *table* feature of a graphing utility to create a table that shows various box heights x and the corresponding volumes V. Use the table to estimate a range of dimensions within which the maximum volume is produced.

(d) Use the graphing utility to graph V and use the range of dimensions from part (c) to find the x-value for which $V(x)$ is maximum.

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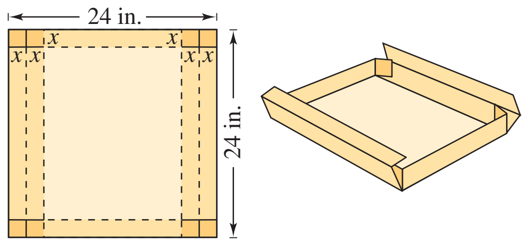
110. **Geometry** An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is done by cutting equal squares from the corners and folding along the dashed lines, as shown in the figure.



(a) Verify that the volume of the box is given by the function $V(x) = 8x(6 - x)(12 - x)$.

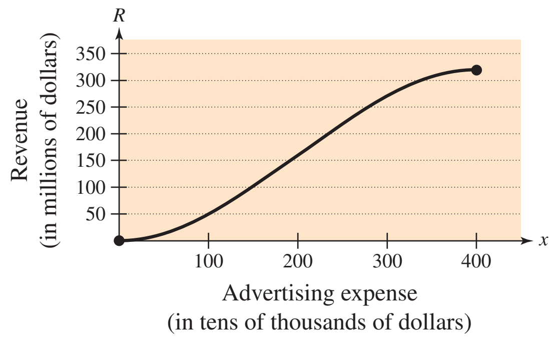
(b) Determine the domain of the function V.

(c) Sketch the graph of the function and estimate the value of x for which $V(x)$ is maximum.

111. **Marketing** The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = 0.00001(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of the function shown in the figure to estimate the point on the graph at which the function is increasing most rapidly. This point is called the **point of diminishing returns** because any expense above this amount will yield less return per dollar invested in advertising.



112. ***Why you should learn it*** *(p. 100)* The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and t $(2 \le t \le 34)$ is its age (in years). Use a graphing utility to graph the function and estimate the age of the tree when it is growing most rapidly. This point is called the **point of diminishing returns** because the increase in growth will be less with each additional year. (*Hint:* Use a viewing window in which $0 \le x \le 35$ and $0 \le y \le 60$.)

113. MODELING DATA

The U.S. production of crude oil y_1 (in quadrillions of British thermal units) and of solar and photovoltaic energy y_2 (in trillions of British thermal units) are shown in the table for the years 1999 through 2008, where t represents the year, with $t = 9$ corresponding to 1999. These data can be approximated by the models

$y_1 = 7.204t^3 - 301.60t^2 + 3854.2t - 3130$ and

$y_2 = 0.077t^3 - 2.31t^2 + 21.3t + 8.$

(Source: Energy Information Administration)

Year, t	y_1	y_2
9	12,451	69
10	12,358	66
11	12,282	65
12	12,163	64
13	12,026	64
14	11,503	65
15	10,963	66
16	10,801	72
17	10,721	81
18	10,519	91

(a) Use a graphing utility to plot the data and graph the model for y_1 in the same viewing window. How closely does the model represent the data?

(b) Extend the viewing window of the graphing utility to show the right-hand behavior of the model y_1. Would you use the model to estimate the production of crude oil in 2010? in 2020? Explain.

(c) Repeat parts (a) and (b) for y_2.

Conclusions

True or False? **In Exercises 114–118, determine whether the statement is true or false. Justify your answer.**

114. It is possible for a sixth-degree polynomial to have only one zero.

115. The graph of the function

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

rises to the left and falls to the right.

116. The graph of the function $f(x) = 2x(x-1)^2(x+3)^3$ crosses the x-axis at $x = 1$.

117. The graph of the function $f(x) = 2x(x-1)^2(x+3)^3$ touches, but does not cross, the x-axis.

118. The graph of the function $f(x) = 2x(x-1)^2(x+3)^3$ rises to the left and falls to the right.

119. Exploration Use a graphing utility to graph

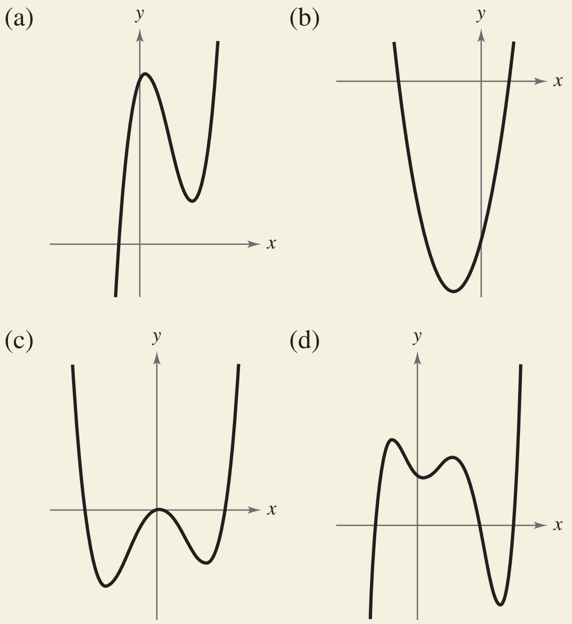
$y_1 = x + 2$ and $y_2 = (x+2)(x-1)$.

Predict the shape of the graph of

$y_3 = (x+2)(x-1)(x-3)$.

Use the graphing utility to verify your answer.

120. CAPSTONE For each graph, describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)



Cumulative Mixed Review

Evaluating Combinations of Functions **In Exercises 121–126, let $f(x) = 14x - 3$ and $g(x) = 8x^2$. Find the indicated value.**

121. $(f + g)(-4)$

122. $(g - f)(3)$

123. $(fg)\left(-\frac{4}{7}\right)$

124. $\left(\frac{f}{g}\right)(-1.5)$

125. $(f \circ g)(-1)$

126. $(g \circ f)(0)$

Solving Inequalities **In Exercises 127–130, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.**

127. $3(x - 5) < 4x - 7$

128. $2x^2 - x \geq 1$

129. $\dfrac{5x - 2}{x - 7} \leq 4$

130. $|x + 8| - 1 \geq 15$

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2.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

Consider the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice in Figure 2.25 that $x = 2$ appears to be a zero of f. Because $f(2) = 0$, you know that $x = 2$ is a zero of the polynomial function f, and that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that $f(x) = (x - 2) \cdot q(x)$. To find $q(x)$, you can use **long division of polynomials.**

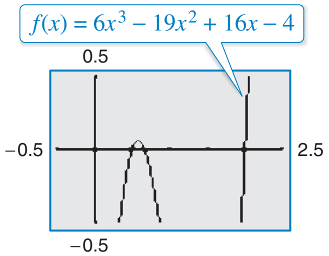

Figure 2.25

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder and Factor Theorems.
- Use the Rational Zero Test to determine possible rational zeros of polynomial functions.
- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Why you should learn it

The Remainder Theorem can be used to determine the number of employees in education and health services in the United States in a given year based on a polynomial model, as shown in Exercise 104 on page 127.

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Example 1 Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Partial quotients

$$
\begin{array}{r@{}r}
 & 6x^2 - 7x + 2 \\
x - 2 \,\big) & \overline{6x^3 - 19x^2 + 16x - 4} \\
 & \underline{6x^3 - 12x^2} \qquad\qquad \\
 & -7x^2 + 16x \qquad \\
 & \underline{-7x^2 + 14x} \qquad \\
 & 2x - 4 \\
 & \underline{2x - 4} \\
 & 0
\end{array}
\qquad
\begin{array}{l}
\\
\\
\text{Multiply: } 6x^2(x - 2). \\
\text{Subtract.} \\
\text{Multiply: } -7x(x - 2). \\
\text{Subtract.} \\
\text{Multiply: } 2(x - 2). \\
\text{Subtract.}
\end{array}
$$

You can see that

$$
\begin{aligned}
6x^3 - 19x^2 + 16x - 4 &= (x - 2)(6x^2 - 7x + 2) \\
&= (x - 2)(2x - 1)(3x - 2).
\end{aligned}
$$

Note that this factorization agrees with the graph of f (see Figure 2.25) in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

CHECKPOINT Now try Exercise 9.

Note that in Example 1, the division process requires $-7x^2 + 14x$ to be subtracted from $-7x^2 + 16x$. Therefore, it is implied that

$$
\begin{array}{r}
-7x^2 + 16x \\
\underline{-(-7x^2 + 14x)}
\end{array}
=
\begin{array}{r}
-7x^2 + 16x \\
\underline{7x^2 - 14x}
\end{array}
$$

and instead is written simply as

$$
\begin{array}{r}
-7x^2 + 16x \\
\underline{-7x^2 + 14x}. \\
2x
\end{array}
$$

In Example 1, $x - 2$ is a factor of the polynomial

$$6x^3 - 19x^2 + 16x - 4$$

and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, when you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$\begin{array}{lrl} & x + 2 & \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow \quad x + 1 & \overline{)\,x^2 + 3x + 5} & \leftarrow \text{Dividend} \\ & \underline{x^2 + x\phantom{{}+5}} & \\ & 2x + 5 & \\ & \underline{2x + 2} & \\ & 3 & \leftarrow \text{Remainder} \end{array}$$

In fractional form, you can write this result as follows.

$$\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{\downarrow \atop 3}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)}\,\underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Note how these steps are applied in the next two examples.

Example 2 Long Division of Polynomials

Divide $8x^3 - 1$ by $2x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$
\begin{array}{r}
4x^2 + 2x + 1 \\
2x - 1 \overline{)\, 8x^3 + 0x^2 + 0x - 1} \\
\underline{8x^3 - 4x^2} \qquad\qquad \\
4x^2 + 0x \qquad \\
\underline{4x^2 - 2x} \qquad \\
2x - 1 \\
\underline{2x - 1} \\
0
\end{array}
$$

So, $2x - 1$ divides evenly into $8x^3 - 1$, and you can write

$$\frac{8x^3 - 1}{2x - 1} = 4x^2 + 2x + 1, \quad x \neq \frac{1}{2}.$$

CHECKPOINT Now try Exercise 15.

You can check the result of Example 2 by multiplying.

$$
\begin{aligned}
(2x - 1)(4x^2 + 2x + 1) &= 8x^3 + 4x^2 + 2x - 4x^2 - 2x - 1 \\
&= 8x^3 - 1
\end{aligned}
$$

In each of the long division examples presented so far, the divisor has been a first-degree polynomial. The long division algorithm works just as well with polynomial divisors of degree two or more, as shown in Example 3.

Example 3 Long Division of Polynomials

Divide $-2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$.

Solution

Begin by writing the dividend in descending powers of x.

$$
\begin{array}{r}
2x^2 \qquad\quad + 1 \\
x^2 + 2x - 3 \overline{)\, 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
\underline{2x^4 + 4x^3 - 6x^2} \qquad\qquad \\
x^2 + 3x - 2 \\
\underline{x^2 + 2x - 3} \\
x + 1
\end{array}
$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

CHECKPOINT Now try Exercise 17.

Synthetic Division

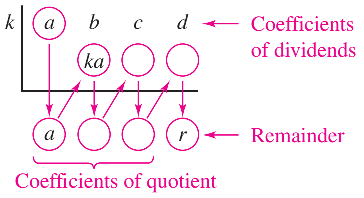
There is a nice shortcut for long division of polynomials when dividing by divisors of the form

$$x - k.$$

The shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

This algorithm for synthetic division works *only* for divisors of the form $x - k$. Remember that

$$x + k = x - (-k).$$

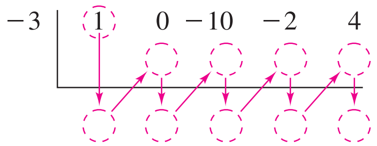
Example 4 Using Synthetic Division

Use synthetic division to divide

$$x^4 - 10x^2 - 2x + 4 \quad \text{by} \quad x + 3.$$

Solution

You should set up the array as follows. Note that a zero is included for each missing term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.

Divisor: $x + 3$

Dividend: $x^4 - 10x^2 - 2x + 4$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$$

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

CHECKPOINT Now try Exercise 23.

Explore the Concept

Evaluate the polynomial $x^4 - 10x^2 - 2x + 4$ at $x = -3$. What do you observe?

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem (See the proof on page 176.)

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

Example 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

CHECKPOINT Now try Exercise 43.

Another important theorem is the **Factor Theorem.** This theorem states that you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, then $(x - k)$ is a factor.

The Factor Theorem (See the proof on page 176.)

A polynomial $f(x)$ has a factor

$$(x - k)$$

if and only if

$$f(k) = 0.$$

Example 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr}
2 & 2 & 7 & -4 & -27 & -18 \\
 & & 4 & 22 & 36 & 18 \\
\hline
 & 2 & 11 & 18 & 9 & 0
\end{array} \longrightarrow$$

0 remainder; $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr}
-3 & 2 & 11 & 18 & 9 \\
 & & -6 & -15 & -9 \\
\hline
 & \underbrace{2 \quad 5 \quad 3}_{2x^2 + 5x + 3} & & & 0
\end{array} \longrightarrow$$

0 remainder; $(x + 3)$ is a factor.

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

CHECKPOINT Now try Exercise 53.

Graphical Solution

From the graph of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$$

you can see that there are four x-intercepts (see Figure 2.26). These occur at $x = -3$, $x = -\frac{3}{2}$, $x = -1$, and $x = 2$. (Check this algebraically.) This implies that $(x + 3)$, $\left(x + \frac{3}{2}\right)$, $(x + 1)$, and $(x - 2)$ are factors of $f(x)$. $\left[\text{Note that } \left(x + \frac{3}{2}\right) \text{ and } (2x + 3)\right.$ are equivalent factors because they both yield the same zero, $\left.x = -\frac{3}{2}.\right]$

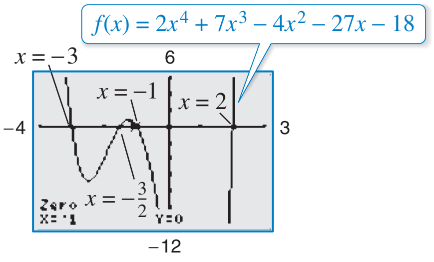


Figure 2.26

Note in Example 6 that the complete factorization of $f(x)$ implies that f has four real zeros:

$$x = 2, \quad x = -3, \quad x = -\tfrac{3}{2}, \quad \text{and} \quad x = -1.$$

This is confirmed by the graph of f, which is shown in Figure 2.26.

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, then $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, then $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, when you find that $x - k$ divides evenly into $f(x)$, try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

> **The Rational Zero Test**
>
> If the polynomial
>
> $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$
>
> has integer coefficients, then every rational zero of f has the form
>
> $$\text{Rational zero} = \frac{p}{q}$$
>
> where p and q have no common factors other than 1, p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Now that you have formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term. This case is illustrated in Example 7.

Example 7 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Possible rational zeros: ± 1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the polynomial has *no* rational zeros. Note from the graph of f in Figure 2.27 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

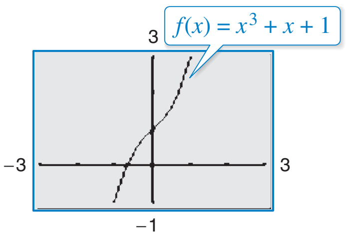


Figure 2.27

CHECKPOINT Now try Exercise 57.

> **Study Tip**
>
> Use a graphing utility to graph the polynomial
>
> $y = x^3 - 53x^2 + 103x - 51$
>
> in a standard viewing window. From the graph alone, it appears that there is only one zero. From the Leading Coefficient Test, you know that because the degree of the polynomial is odd and the leading coefficient is positive, the graph falls to the left and rises to the right. So, the function must have another zero. From the Rational Zero Test, you know that ± 51 might be zeros of the function. When you zoom out several times, you will see a more complete picture of the graph. Your graph should confirm that $x = 51$ is a zero of f.

When the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

1. A programmable calculator can be used to speed up the calculations.
2. A graphing utility can give a good estimate of the locations of the zeros.
3. The Intermediate Value Theorem, along with a table generated by a graphing utility, can give approximations of zeros.
4. The Factor Theorem and synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division, as shown in Example 8.

Example 8 Using the Rational Zero Test

Find the rational zeros of

$$f(x) = 2x^3 + 3x^2 - 8x + 3.$$

Solution

The leading coefficient is 2 and the constant term is 3.

Possible rational zeros:

$$\frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in Figure 2.28.

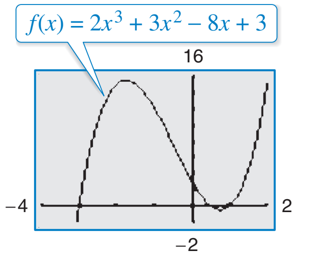


Figure 2.28

CHECKPOINT Now try Exercise 59.

Remember that when you try to find the rational zeros of a polynomial function with many possible rational zeros, as in Example 8, you must use trial and error. There is no quick algebraic method to determine which of the possibilities is an actual zero; however, sketching a graph may be helpful.

Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Decartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive (nonzero) coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 9 Using Descartes's Rule of Signs

Describe the possible real zeros of $f(x) = 3x^3 - 5x^2 + 6x - 4$.

Solution

The original polynomial has *three* variations in sign.

+ to − + to −

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

− to +

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4$$
$$= -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. By using the *trace* feature of a graphing utility, you can see that the function has only one real zero (it is a positive number near $x = 1$), as shown in Figure 2.29.

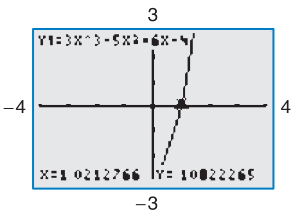


Figure 2.29

CHECKPOINT Now try Exercise 61.

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound of the real zeros of f, which can help you eliminate possible real zeros. A real number c is an **upper bound** for the real zeros of f when no zeros are greater than c. Similarly, c is a **lower bound** when no real zeros of f are less than c.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), then c is a **lower bound** for the real zeros of f.

Example 10 Finding the Zeros of a Polynomial Function

Find the real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of } 2}{\text{Factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$f(-x) = 6(-x)^3 - 4(-x)^2 + 3(-x) - 2$$
$$= -6x^3 - 4x^2 - 3x - 2$$

has no variations in sign. As a result of these two findings, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative real zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. Therefore, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero, as shown in Figure 2.30.

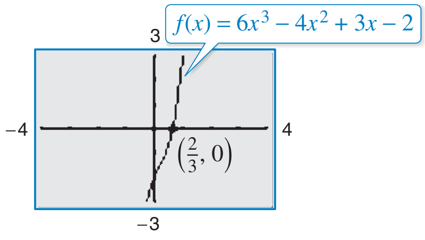


Figure 2.30

CHECKPOINT Now try Exercise 71.

Explore the Concept

Use a graphing utility to graph the polynomial

$$y_1 = 6x^3 - 4x^2 + 3x - 2.$$

Notice that the graph intersects the x-axis at the point $\left(\frac{2}{3}, 0\right)$. How does this information relate to the real zero found in Example 10? Use the graphing utility to graph

$$y_2 = x^4 - 5x^3 + 3x^2 + x.$$

How many times does the graph intersect the x-axis? How many real zeros does y_2 have?

Here are two additional hints that can help you find the real zeros of a polynomial.

1. When the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. When you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, after writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x - 1)(x^2 - 4x - 1)$$

you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Note how these hints are applied in the next example.

Example 11 Finding the Zeros of a Polynomial Function

Find all the real zeros of $f(x) = 10x^4 - 15x^3 - 16x^2 + 12x$.

Solution

Remove the common monomial factor x to write

$$f(x) = 10x^4 - 15x^3 - 16x^2 + 12x = x(10x^3 - 15x^2 - 16x + 12).$$

So, $x = 0$ is a zero of f. You can find the remaining zeros of f by analyzing the cubic factor. Because the leading coefficient is 10 and the constant term is 12, there is a long list of possible rational zeros.

Possible rational zeros:

$$\frac{\text{Factors of } 12}{\text{Factors of } 10} = \frac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm12}{\pm1, \pm2, \pm5, \pm10}$$

With so many possibilities (32, in fact), it is worth your time to use a graphing utility to focus on just a few. By using the *trace* feature of a graphing utility, it looks like three reasonable choices are $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$ (see Figure 2.31). Synthetic division shows that only $x = 2$ works. (You could also use the Factor Theorem to test these choices.)

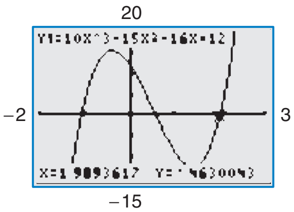


Figure 2.31

$$\begin{array}{r|rrrr} 2 & 10 & -15 & -16 & 12 \\ & & 20 & 10 & -12 \\ \hline & 10 & 5 & -6 & 0 \end{array}$$

So, $x = 2$ is one zero and you have

$$f(x) = x(x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.56 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.06$$

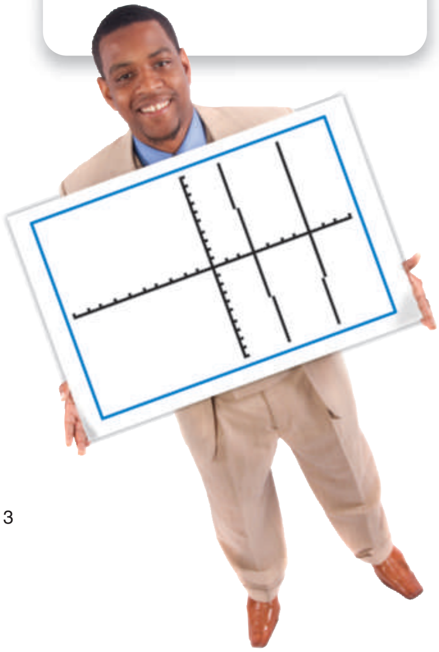
CHECKPOINT Now try Exercise 87.

Explore the Concept

Use a graphing utility to graph the polynomial

$$y = x^3 + 4.8x^2 - 127x + 309$$

in a standard viewing window. From the graph, what do the real zeros appear to be? Discuss how the mathematical tools of this section might help you realize that the graph does not show all the important features of the polynomial function. Now use the *zoom* feature to find all the zeros of this function.



Bryan Creely/iStockphoto.com

2.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Two forms of the Division Algorithm are shown below. Identify and label each part.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–5, fill in the blank(s).

2. The rational expression $p(x)/q(x)$ is called _______ when the degree of the numerator is greater than or equal to that of the denominator.

3. Every rational zero of a polynomial function with integer coefficients has the form p/q, where p is a factor of the _______ and q is a factor of the _______.

4. The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called _______ of _______ .

5. A real number c is a(n) _______ bound for the real zeros of f when no zeros are greater than c, and is a(n) _______ bound when no real zeros of f are less than c.

6. How many negative real zeros are possible for a polynomial function $f(x)$, given that $f(-x)$ has 5 variations in sign?

7. You divide the polynomial $f(x)$ by $(x - 4)$ and obtain a remainder of 7. What is $f(4)$?

8. What value should you write in the circle to check whether $(x - 3)$ is a factor of $f(x) = x^3 - 2x^2 + 3x + 4$?

$$\bigcirc \;\; \underline{\big|\; 1 \quad -2 \quad 3 \quad 4}$$

Procedures and Problem Solving

Long Division of Polynomials **In Exercises 9–22, use long division to divide.**

9. Divide $2x^2 + 10x + 12$ by $x + 3$.

10. Divide $5x^2 - 17x - 12$ by $x - 4$.

11. Divide $x^4 + 5x^3 + 6x^2 - x - 2$ by $x + 2$.

12. Divide $x^3 - 4x^2 - 17x + 6$ by $x - 3$.

13. Divide $4x^3 - 7x^2 - 11x + 5$ by $4x + 5$.

14. Divide $2x^3 - 3x^2 - 50x + 75$ by $2x - 3$.

15. Divide $7x^3 + 3$ by $x + 2$.

16. Divide $8x^4 - 5$ by $2x + 1$.

17. $(x + 8 + 6x^3 + 10x^2) \div (2x^2 + 1)$

18. $(1 + 3x^2 + x^4) \div (3 - 2x + x^2)$

19. $(x^3 - 9) \div (x^2 + 1)$

20. $(x^5 + 7) \div (x^3 - 1)$

21. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x - 1)^2}$

22. $\dfrac{x^4}{(x - 1)^3}$

Using Synthetic Division **In Exercises 23–32, use synthetic division to divide.**

23. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

24. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

25. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$

26. $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$

27. $(9x^3 - 18x^2 - 16x + 32) \div (x - 2)$

28. $(5x^3 + 6x + 8) \div (x + 2)$

29. $(x^3 + 512) \div (x + 8)$

30. $(x^3 - 729) \div (x - 9)$

31. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$

32. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

Verifying Quotients **In Exercises 33–36, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify the results algebraically.**

33. $y_1 = \dfrac{x^2}{x + 2}, \quad y_2 = x - 2 + \dfrac{4}{x + 2}$

34. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}, \quad y_2 = x - 1 + \dfrac{2}{x + 3}$

35. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

36. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2 + 1}, \quad y_2 = x^2 - \dfrac{1}{x^2 + 1}$

Verifying the Remainder Theorem **In Exercises 37–42, write the function in the form $f(x) = (x - k)q(x) + r(x)$ for the given value of k. Use a graphing utility to demonstrate that $f(k) = r$.**

	Function	*Value of k*
37.	$f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
38.	$f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$

	Function	*Value of k*
39.	$f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
40.	$f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
41.	$f(x) = 4x^3 - 6x^2 - 12x - 4$	$k = 1 - \sqrt{3}$
42.	$f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

Using the Remainder Theorem **In Exercises 43–46, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.**

✓ **43.** $f(x) = 2x^3 - 7x + 3$
(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

44. $g(x) = 2x^6 + 3x^4 - x^2 + 3$
(a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

45. $h(x) = x^3 - 5x^2 - 7x + 4$
(a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

46. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$
(a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

Using the Factor Theorem **In Exercises 47–50, use synthetic division to show that *x* is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real solutions of the equation.**

	Polynomial Equation	*Value of x*
47.	$x^3 - 7x + 6 = 0$	$x = 2$
48.	$x^3 - 28x - 48 = 0$	$x = -4$
49.	$2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
50.	$48x^3 - 80x^2 + 41x - 6 = 0$	$x = \frac{2}{3}$

Factoring a Polynomial **In Exercises 51–56, (a) verify the given factor(s) of the function *f*, (b) find the remaining factors of *f*, (c) use your results to write the complete factorization of *f*, and (d) list all real zeros of *f*. Confirm your results by using a graphing utility to graph the function.**

	Function	*Factor(s)*
51.	$f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2)$
52.	$f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3)$
✓ **53.**	$f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
54.	$f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
55.	$f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1)$
56.	$f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1)$

Using the Rational Zero Test **In Exercises 57–60, use the Rational Zero Test to list all possible rational zeros of *f*. Then find the rational zeros.**

✓ **57.** $f(x) = x^3 + 3x^2 - x - 3$

58. $f(x) = x^3 - 4x^2 - 4x + 16$

✓ **59.** $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$

60. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$

Using Descartes's Rule of Signs **In Exercises 61–64, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.**

✓ **61.** $f(x) = 2x^4 - x^3 + 6x^2 - x + 5$

62. $f(x) = 3x^4 + 5x^3 - 6x^2 + 8x - 3$

63. $g(x) = 4x^3 - 5x + 8$

64. $g(x) = 2x^3 - 4x^2 - 5$

Finding the Zeros of a Polynomial Function **In Exercises 65–70, (a) use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of *f*, (b) list the possible rational zeros of *f*, (c) use a graphing utility to graph *f* so that some of the possible zeros in parts (a) and (b) can be disregarded, and (d) determine all the real zeros of *f*.**

65. $f(x) = x^3 + x^2 - 4x - 4$

66. $f(x) = -3x^3 + 20x^2 - 36x + 16$

67. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

68. $f(x) = 4x^4 - 17x^2 + 4$

69. $f(x) = 32x^3 - 52x^2 + 17x + 3$

70. $f(x) = x^4 - x^3 - 29x^2 - x - 30$

Finding the Zeros of a Polynomial Function **In Exercises 71–74, use synthetic division to verify the upper and lower bounds of the real zeros of *f*. Then find the real zeros of the function.**

✓ **71.** $f(x) = x^4 - 4x^3 + 15$
Upper bound: $x = 4$
Lower bound: $x = -1$

72. $f(x) = 2x^3 - 3x^2 - 12x + 8$
Upper bound: $x = 4$
Lower bound: $x = -3$

73. $f(x) = x^4 - 4x^3 + 16x - 16$
Upper bound: $x = 5$
Lower bound: $x = -3$

74. $f(x) = 2x^4 - 8x + 3$
Upper bound: $x = 3$
Lower bound: $x = -4$

Occasionally, throughout this text, you will be asked to round to a place value rather than to a number of decimal places.

Rewriting to Use the Rational Zero Test In Exercises 75–78, find the rational zeros of the polynomial function.

75. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

76. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

77. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

78. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

A Cubic Polynomial with Two Terms In Exercises 79–82, match the cubic function with the correct number of rational and irrational zeros.

(a) Rational zeros: 0; Irrational zeros: 1

(b) Rational zeros: 3; Irrational zeros: 0

(c) Rational zeros: 1; Irrational zeros: 2

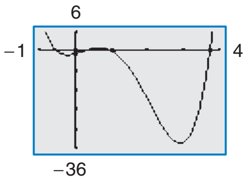
(d) Rational zeros: 1; Irrational zeros: 0

79. $f(x) = x^3 - 1$ **80.** $f(x) = x^3 - 2$

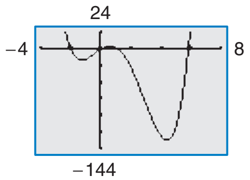
81. $f(x) = x^3 - x$ **82.** $f(x) = x^3 - 2x$

Using a Graph to Help Find Zeros In Exercises 83–86, the graph of $y = f(x)$ is shown. Use the graph as an aid to find all the real zeros of the function.

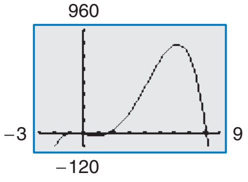
83. $y = 2x^4 - 9x^3 + 5x^2 + 3x - 1$



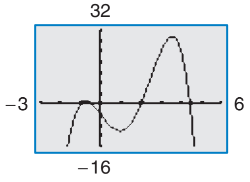
84. $y = x^4 - 5x^3 - 7x^2 + 13x - 2$



85. $y = -2x^4 + 17x^3 - 3x^2 - 25x - 3$



86. $y = -x^4 + 5x^3 - 10x - 4$



Finding the Zeros of a Polynomial Function In Exercises 87–98, find all real zeros of the polynomial function.

87. $f(x) = 3x^4 - 14x^2 - 4x$

88. $g(x) = 4x^4 - 11x^3 - 22x^2 + 8x$

89. $f(z) = z^4 - z^3 - 2z - 4$

90. $f(x) = 4x^3 + 7x^2 - 11x - 18$

91. $g(y) = 2y^4 + 7y^3 - 26y^2 + 23y - 6$

92. $h(x) = x^5 - x^4 - 3x^3 + 5x^2 - 2x$

93. $f(x) = 4x^4 - 55x^2 - 45x + 36$

94. $z(x) = 4x^4 - 43x^2 - 9x + 90$

95. $g(x) = 8x^4 + 28x^3 + 9x^2 - 9x$

96. $h(x) = x^5 + 5x^4 - 5x^3 - 15x^2 - 6x$

97. $f(x) = 4x^5 + 12x^4 - 11x^3 - 42x^2 + 7x + 30$

98. $g(x) = 4x^5 + 8x^4 - 15x^3 - 23x^2 + 11x + 15$

Using a Rational Zero In Exercises 99–102, (a) use the *zero* or *root* feature of a graphing utility to approximate (accurate to the nearest thousandth) the zeros of the function, (b) determine one of the exact zeros and use synthetic division to verify your result, and (c) factor the polynomial completely.

99. $h(t) = t^3 - 2t^2 - 7t + 2$

100. $f(s) = s^3 - 12s^2 + 40s - 24$

101. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

102. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

103. MODELING DATA

The table shows the numbers S of cellular phone subscriptions per 100 people in the United States from 1991 through 2008. (Source: U.S. International Telecommunications Union)

Year	Subscriptions per 100 people, S
1991	3.0
1992	4.3
1993	6.2
1994	9.2
1995	12.7
1996	16.4
1997	20.3
1998	25.1
1999	30.8
2000	38.9
2001	45.1
2002	49.2
2003	55.2
2004	62.9
2005	71.5
2006	77.4
2007	85.2
2008	86.8

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The data can be approximated by the model

$S = -0.0135t^3 + 0.545t^2 - 0.71t + 3.6, \quad 1 \le t \le 18$

where t represents the year, with $t = 1$ corresponding to 1991.

(a) Use a graphing utility to graph the data and the model in the same viewing window.

(b) How well does the model fit the data?

(c) Use the Remainder Theorem to evaluate the model for the year 2015. Is the value reasonable? Explain.

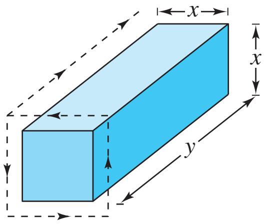
104. ***Why you should learn it*** *(p. 113)* The numbers of employees E (in thousands) in education and health services in the United States from 1960 through 2008 are approximated by $E = -0.084t^3 + 10.32t^2 - 23.5t + 3167$, $0 \le t \le 48$, where t is the year, with $t = 0$ corresponding to 1960. (Source: U.S. Bureau of Labor Statistics)

(a) Use a graphing utility to graph the model over the domain.

(b) Estimate the number of employees in education and health services in 1960. Use the Remainder Theorem to estimate the number in 2000.

(c) Is this a good model for making predictions in future years? Explain.

105. Geometry A rectangular package sent by a delivery service can have a maximum combined length and girth (perimeter of a cross section) of 120 inches (see figure).



(a) Show that the volume of the package is given by the function $V(x) = 4x^2(30 - x)$.

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.

106. Environmental Science The number of parts per million of nitric oxide emissions y from a car engine is approximated by $y = -5.05x^3 + 3857x - 38{,}411.25$, $13 \le x \le 18$, where x is the air-fuel ratio.

(a) Use a graphing utility to graph the model.

(b) There are two air-fuel ratios that produce 2400 parts per million of nitric oxide. One is $x = 15$. Use the graph to approximate the other.

(c) Find the second air-fuel ratio from part (b) algebraically. (*Hint:* Use the known value of $x = 15$ and synthetic division.)

Conclusions

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

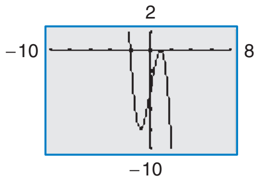
107. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

108. The value $x = \frac{1}{7}$ is a zero of the polynomial function

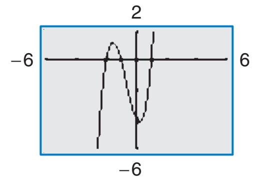
$$f(x) = 3x^5 - 2x^4 + x^3 - 16x^2 + 3x - 8.$$

Think About It **In Exercises 109 and 110, the graph of a cubic polynomial function $y = f(x)$ with integer zeros is shown. Find the factored form of f.**

109.



110.



111. Think About It Let $y = f(x)$ be a fourth-degree polynomial with leading coefficient $a = -1$ and $f(\pm 1) = f(\pm 2) = 0$. Find the factored form of f.

112. Think About It Find the value of k such that $x - 3$ is a factor of $x^3 - kx^2 + 2kx - 12$.

113. Writing Complete each polynomial division. Write a brief description of the pattern that you obtain, and use your result to find a formula for the polynomial division $(x^n - 1)/(x - 1)$. Create a numerical example to test your formula.

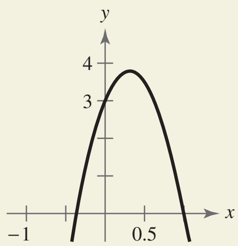
(a) $\dfrac{x^2 - 1}{x - 1} = \square$ (b) $\dfrac{x^3 - 1}{x - 1} = \square$

(c) $\dfrac{x^4 - 1}{x - 1} = \square$

114. CAPSTONE A graph of $f(x)$ is shown, where

$f(x) = 2x^5 - 3x^4 + x^3 - 8x^2 + 5x + 3$ and

$f(-x) = -2x^5 - 3x^4 - x^3 - 8x^2 - 5x + 3$.



(a) How many negative real zeros does f have? Explain.

(b) How many positive real zeros are *possible* for f? Explain. What does this tell you about the eventual right-hand behavior of the graph?

(c) Is $x = -\frac{1}{3}$ a possible rational zero of f? Explain.

(d) Explain how to check whether $\left(x - \frac{3}{2}\right)$ is a factor of f and whether $x = \frac{3}{2}$ is an upper bound for the real zeros of f.

Cumulative Mixed Review

Solving a Quadratic Equation **In Exercises 115–118, use any convenient method to solve the quadratic equation.**

115. $9x^2 - 25 = 0$ **116.** $16x^2 - 21 = 0$

117. $2x^2 + 6x + 3 = 0$ **118.** $8x^2 - 22x + 15 = 0$

2.4 Complex Numbers

The Imaginary Unit *i*

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\begin{aligned}\sqrt{-9} - 5 &= \sqrt{3^2(-1)} - 5\\ &= 3\sqrt{-1} - 5\\ &= 3i - 5\\ &= -5 + 3i.\end{aligned}$$

In the standard form $a + bi$, the real number a is called the **real part** of the **complex number** $a + bi$, and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

If a and b are real numbers, then the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, then the number $a + bi = a$ is a real number. If $b \neq 0$, then the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.32. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

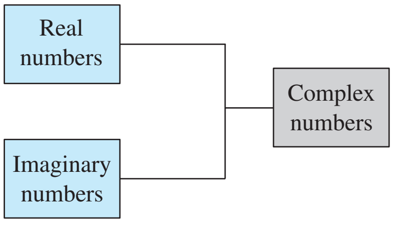


Figure 2.32

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

What you should learn

- Use the imaginary unit *i* to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Find complex solutions of quadratic equations.

Why you should learn it

Complex numbers are used to model numerous aspects of the natural world, such as the impedance of an electrical circuit, as shown in Exercise 94 on page 134.

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Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, then their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$-(a + bi) = -a - bi.$ Additive inverse

So, you have

$(a + bi) + (-a - bi) = 0 + 0i = 0.$

Example 1 Adding and Subtracting Complex Numbers

Perform the operation(s) and write the result in standard form.

a. $(3 - i) + (2 + 3i)$ **b.** $(1 + 2i) - (4 + 2i)$

c. $3 - (-2 + 3i) + (-5 + i)$ **d.** $(3 + 2i) + (4 - i) - (7 + i)$

Solution

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ Remove parentheses.

$= (3 + 2) + (-i + 3i)$ Group like terms.

$= 5 + 2i$ Write in standard form.

b. $(1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i$ Remove parentheses.

$= (1 - 4) + (2i - 2i)$ Group like terms.

$= -3 + 0i$ Simplify.

$= -3$ Write in standard form.

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$= (3 + 2 - 5) + (-3i + i)$

$= 0 - 2i$

$= -2i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2i - i - i)$

$= 0 + 0i$

$= 0$

CHECKPOINT Now try Exercise 25.

In Examples 1(b) and 1(d), note that the sum of complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication
Commutative Properties of Addition and Multiplication
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$(a + bi)(c + di) = a(c + di) + bi(c + di) \quad \text{Distributive Property}$$
$$= ac + (ad)i + (bc)i + (bd)i^2 \quad \text{Distributive Property}$$
$$= ac + (ad)i + (bc)i + (bd)(-1) \quad i^2 = -1$$
$$= ac - bd + (ad)i + (bc)i \quad \text{Commutative Property}$$
$$= (ac - bd) + (ad + bc)i \quad \text{Associative Property}$$

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method.

Explore the Concept

Complete the following:

$i^1 = i$	$i^7 =$
$i^2 = -1$	$i^8 =$
$i^3 = -i$	$i^9 =$
$i^4 = 1$	$i^{10} =$
$i^5 =$	$i^{11} =$
$i^6 =$	$i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

Example 2 Multiplying Complex Numbers

Perform the operation(s) and write the result in standard form.

a. $5(-2 + 3i)$
b. $(2 - i)(4 + 3i)$
c. $(3 + 2i)(3 - 2i)$
d. $4i(-1 + 5i)$
e. $(3 + 2i)^2$

Solution

a. $5(-2 + 3i) = 5(-2) + 5(3i)$ — Distributive Property
$= -10 + 15i$ — Simplify.

b. $(2 - i)(4 + 3i) = 2(4 + 3i) - i(4 + 3i)$ — Distributive Property
$= 8 + 6i - 4i - 3i^2$ — Product of binomials
$= 8 + 6i - 4i - 3(-1)$ — $i^2 = -1$
$= 8 + 3 + 6i - 4i$ — Group like terms.
$= 11 + 2i$ — Write in standard form.

c. $(3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)$ — Distributive Property
$= 9 - 6i + 6i - 4i^2$ — Product of binomials
$= 9 - 4(-1)$ — $i^2 = -1$
$= 9 + 4$ — Simplify.
$= 13$ — Write in standard form.

d. $4i(-1 + 5i) = 4i(-1) + 4i(5i)$ — Distributive Property
$= -4i + 20i^2$ — Simplify.
$= -4i + 20(-1)$ — $i^2 = -1$
$= -20 - 4i$ — Write in standard form.

e. $(3 + 2i)^2 = 9 + 6i + 6i + 4i^2$ — Product of binomials
$= 9 + 12i + 4(-1)$ — $i^2 = -1$
$= 9 - 4 + 12i$ — Group like terms.
$= 5 + 12i$ — Write in standard form.

CHECKPOINT Now try Exercise 33.

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the forms $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

Example 3 Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $3 - 5i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $3 - 5i$ is $3 + 5i$.

$$(3 - 5i)(3 + 5i) = 3^2 - (5i)^2 = 9 - 25i^2 = 9 - 25(-1) = 34$$

CHECKPOINT Now try Exercise 43.

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) \quad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i. \quad \text{Standard form}$$

Example 4 Writing a Quotient of Complex Numbers in Standard Form

Write the quotient $\dfrac{2 + 3i}{4 - 2i}$ in standard form.

Solution

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) \quad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} \quad \text{Expand.}$$

$$= \frac{8 - 6 + 16i}{16 + 4} \quad i^2 = -1$$

$$= \frac{2 + 16i}{20} \quad \text{Simplify.}$$

$$= \frac{1}{10} + \frac{4}{5}i \quad \text{Write in standard form.}$$

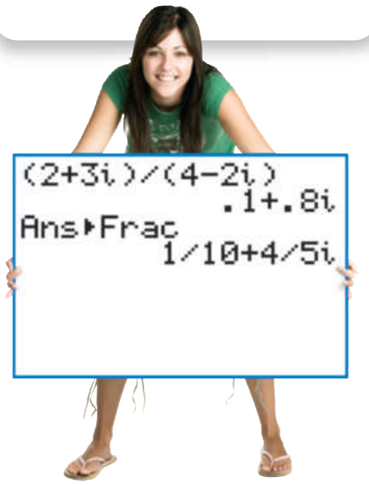
CHECKPOINT Now try Exercise 55.

Technology Tip

Some graphing utilities can perform operations with complex numbers. For instance, on some graphing utilities, to divide $2 + 3i$ by $4 - 2i$, use the following keystrokes.

(2 + 3 i) ÷

(4 − 2 i) ENTER



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Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}\,i$$

The number $\sqrt{3}\,i$ is called the *principal square root* of -3.

Principal Square Root of a Negative Number

If a is a positive number, then the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}\,i.$$

Example 5 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}\,i\sqrt{12}\,i$
$= \sqrt{36}\,i^2$
$= 6(-1)$
$= -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}\,i - \sqrt{27}\,i$
$= 4\sqrt{3}\,i - 3\sqrt{3}\,i$
$= \sqrt{3}\,i$

c. $(-1 + \sqrt{-3})^2 = (-1 + \sqrt{3}\,i)^2$
$= (-1)^2 - 2\sqrt{3}\,i + (\sqrt{3})^2(i^2)$
$= 1 - 2\sqrt{3}\,i + 3(-1)$
$= -2 - 2\sqrt{3}\,i$

CHECKPOINT Now try Exercise 63.

Example 6 Complex Solutions of a Quadratic Equation

Solve (a) $x^2 + 4 = 0$ and (b) $3x^2 - 2x + 5 = 0$.

Solution

a. $x^2 + 4 = 0$ Write original equation.
$x^2 = -4$ Subtract 4 from each side.
$x = \pm 2i$ Extract square roots.

b. $3x^2 - 2x + 5 = 0$ Write original equation.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$ Quadratic Formula

$$= \frac{2 \pm \sqrt{-56}}{6}$$ Simplify.

$$= \frac{2 \pm 2\sqrt{14}\,i}{6}$$ Write $\sqrt{-56}$ in standard form.

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$ Write in standard form.

CHECKPOINT Now try Exercise 73.

Study Tip

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid when *both* a and b are negative. For example,

$$\sqrt{-5}\sqrt{-5} = \sqrt{5(-1)}\sqrt{5(-1)}$$
$$= \sqrt{5}\,i\sqrt{5}\,i$$
$$= \sqrt{25}\,i^2$$
$$= 5i^2 = -5$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

2.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match the type of complex number with its definition.

(a) real number (i) $a + bi, a = 0, b \neq 0$

(b) imaginary number (ii) $a + bi, b = 0$

(c) pure imaginary number (iii) $a + bi, a \neq 0, b \neq 0$

In Exercises 2 and 3, fill in the blanks.

2. The imaginary unit i is defined as $i =$ _______ , where $i^2 =$ _______ .

3. When you add $(7 + 6i)$ and $(8 + 5i)$, the real part of the sum is _______ and the imaginary part of the sum is _______.

4. What method for multiplying two polynomials can you use when multiplying two complex numbers?

5. What is the additive inverse of the complex number $2 - 4i$?

6. What is the complex conjugate of the complex number $2 - 4i$?

Procedures and Problem Solving

Equality of Complex Numbers **In Exercises 7–10, find real numbers a and b such that the equation is true.**

7. $a + bi = -9 + 4i$

8. $a + bi = 12 + 5i$

9. $3a + (b + 3)i = 9 + 8i$

10. $(a + 6) + 2bi = 6 - i$

Writing a Complex Number in Standard Form **In Exercises 11–20, write the complex number in standard form.**

11. $5 + \sqrt{-16}$

12. $2 - \sqrt{-9}$

13. -6

14. 8

15. $-5i + i^2$

16. $-3i^2 + i$

17. $\left(\sqrt{-75}\right)^2$

18. $\left(\sqrt{-4}\right)^2 - 7$

19. $\sqrt{-0.09}$

20. $\sqrt{-0.0004}$

Adding and Subtracting Complex Numbers **In Exercises 21–30, perform the addition or subtraction and write the result in standard form.**

21. $(4 + i) - (7 - 2i)$

22. $(11 - 2i) - (-3 + 6i)$

23. $(-1 + 8i) + (8 - 5i)$

24. $(7 + 6i) + (3 + 12i)$

25. $13i - (14 - 7i)$

26. $22 + (-5 + 8i) - 9i$

27. $\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$

28. $\left(\frac{3}{4} + \frac{7}{5}i\right) - \left(\frac{5}{6} - \frac{1}{6}i\right)$

29. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

30. $-(-3.7 - 12.8i) - (6.1 - 16.3i)$

Multiplying Complex Numbers **In Exercises 31–42, perform the operation and write the result in standard form.**

31. $4(3 + 5i)$

32. $-6(5 - 3i)$

33. $(1 + i)(3 - 2i)$

34. $(6 - 2i)(2 - 3i)$

35. $4i(8 + 5i)$

36. $-3i(6 - i)$

37. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

38. $\left(\sqrt{3} + \sqrt{15}i\right)\left(\sqrt{3} - \sqrt{15}i\right)$

39. $(6 + 7i)^2$

40. $(5 - 4i)^2$

41. $(4 + 5i)^2 - (4 - 5i)^2$

42. $(1 - 2i)^2 - (1 + 2i)^2$

Multiplying Conjugates **In Exercises 43–50, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.**

43. $4 + 3i$

44. $7 - 5i$

45. $-6 - \sqrt{5}i$

46. $-3 + \sqrt{2}i$

47. $\sqrt{-20}$

48. $\sqrt{-13}$

49. $3 - \sqrt{-2}$

50. $1 + \sqrt{-8}$

Writing a Quotient of Complex Numbers in Standard Form **In Exercises 51–58, write the quotient in standard form.**

51. $\dfrac{6}{i}$

52. $-\dfrac{5}{2i}$

53. $\dfrac{2}{4 - 5i}$

54. $\dfrac{3}{1 - i}$

55. $\dfrac{2 + i}{2 - i}$

56. $\dfrac{8 - 7i}{1 - 2i}$

57. $i/(4 - 5i)^2$

58. $5i/(2 + 3i)^2$

Adding or Subtracting Quotients of Complex Numbers **In Exercises 59–62, perform the operation and write the result in standard form.**

59. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$

60. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

61. $\dfrac{i}{3-2i}+\dfrac{2i}{3+8i}$ **62.** $\dfrac{1+i}{i}-\dfrac{3}{4-i}$

Writing Complex Numbers in Standard Form **In Exercises 63–72, perform the operation and write the result in standard form.**

63. $\sqrt{-18}-\sqrt{-54}$ **64.** $\sqrt{-50}+\sqrt{-275}$

65. $\left(-3+\sqrt{-24}\right)+\left(7-\sqrt{-44}\right)$

66. $\left(-12-\sqrt{-72}\right)+\left(9+\sqrt{-108}\right)$

67. $\sqrt{-6}\cdot\sqrt{-2}$ **68.** $\sqrt{-5}\cdot\sqrt{-10}$

69. $\left(\sqrt{-10}\right)^2$ **70.** $\left(\sqrt{-75}\right)^2$

71. $\left(2-\sqrt{-6}\right)^2$ **72.** $\left(3+\sqrt{-5}\right)\left(7-\sqrt{-10}\right)$

Complex Solutions of a Quadratic Equation **In Exercises 73–84, solve the quadratic equation.**

73. $x^2+25=0$ **74.** $x^2+32=0$

75. $x^2-2x+2=0$ **76.** $x^2+6x+10=0$

77. $4x^2+16x+17=0$ **78.** $9x^2-6x+37=0$

79. $16t^2-4t+3=0$ **80.** $4x^2+16x+15=0$

81. $\frac{3}{2}x^2-6x+9=0$ **82.** $\frac{7}{8}x^2-\frac{3}{4}x+\frac{5}{16}=0$

83. $1.4x^2-2x-10=0$ **84.** $4.5x^2-3x+12=0$

Expressions Involving Powers of *i* **In Exercises 85–90, simplify the complex number and write it in standard form.**

85. $-6i^3+i^2$ **86.** $4i^2-2i^3$

87. $\left(\sqrt{-75}\right)^3$ **88.** $\left(\sqrt{-2}\right)^6$

89. $\dfrac{1}{i^3}$ **90.** $\dfrac{1}{(2i)^3}$

91. Cube each complex number. What do you notice?

(a) 2 (b) $-1+\sqrt{3}i$ (c) $-1-\sqrt{3}i$

92. Raise each complex number to the fourth power and simplify.

(a) 2 (b) -2 (c) $2i$ (d) $-2i$

93. Use the results of the Explore the Concept on page 130 to find each power of i.

(a) i^{20} (b) i^{45} (c) i^{67} (d) i^{114}

94. ***Why you should learn it*** *(p. 128)* The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation $1/z = 1/z_1 + 1/z_2$, where z_1 is the impedance (in ohms) of pathway 1, and z_2 is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of each parallel circuit. (*Hint:* You can find the impedance of each pathway in a parallel circuit by adding the impedances of all components in the pathway.)

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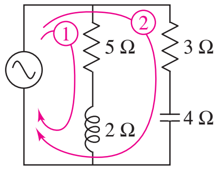
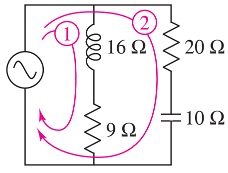
	Resistor	Inductor	Capacitor
Symbol	$a\ \Omega$	$b\ \Omega$	$c\ \Omega$
Impedance	a	bi	$-ci$

(a)



(b)



Conclusions

True or False? **In Exercises 95–100, determine whether the statement is true or false. Justify your answer.**

95. No complex number is equal to its complex conjugate.

96. $i^{44}+i^{150}-i^{74}-i^{109}+i^{61}=-1$

97. The sum of two imaginary numbers is always an imaginary number.

98. The product of two imaginary numbers is always an imaginary number.

99. The conjugate of the product of two complex numbers is equal to the product of the conjugates of the two complex numbers.

100. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.

101. **Error Analysis** Describe the error.

$$\cancel{\sqrt{-6}\sqrt{-6}=\sqrt{(-6)(-6)}=\sqrt{36}=6}$$

102. **CAPSTONE** Consider the functions

$f(x)=2(x-3)^2-4$ and $g(x)=-2(x-3)^2-4$.

(a) Without graphing either function, determine whether the graph of f and the graph of g have x-intercepts. Explain your reasoning.

(b) Solve $f(x)=0$ and $g(x)=0$.

(c) Explain how the zeros of f and g are related to whether their graphs have x-intercepts.

(d) For the function $f(x)=a(x-h)^2+k$, make a general statement about how a, h, and k affect whether the graph of f has x-intercepts, and whether the zeros of f are real or complex.

Cumulative Mixed Review

Multiplying Polynomials **In Exercises 103–106, perform the operation and write the result in standard form.**

103. $(4x-5)(4x+5)$ **104.** $(x+2)^3$

105. $\left(3x-\frac{1}{2}\right)(x+4)$ **106.** $(2x-5)^2$

2.5 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n zeros*. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem (See the proof on page 177.)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, . . . , c_n$ are complex numbers.

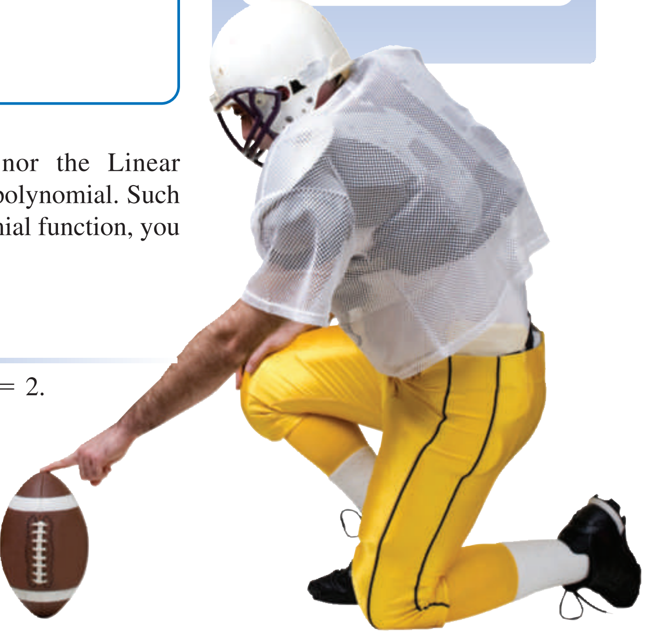
Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques.

What you should learn

- Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function.
- Find all zeros of polynomial functions, including complex zeros.
- Find conjugate pairs of complex zeros.
- Find zeros of polynomials by factoring.

Why you should learn it

Being able to find zeros of polynomial functions is an important part of modeling real-life problems. For instance, Exercise 71 on page 141 shows how to determine whether a football kicked with a given velocity can reach a certain height.



Example 1 Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9$$
$$= (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function

$$f(x) = x^4 - 1$$
$$= (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

CHECKPOINT Now try Exercise 5.

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Finding Zeros of a Polynomial Function

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Examples 2 and 3 illustrate several cases.

Example 2 Real and Complex Zeros of a Polynomial Function

Confirm that the third-degree polynomial function

$$f(x) = x^3 + 4x$$

has exactly three zeros: $x = 0$, $x = 2i$, and $x = -2i$.

Solution

Factor the polynomial completely as $x(x - 2i)(x + 2i)$. So, the zeros are

$$x(x - 2i)(x + 2i) = 0$$

$$x = 0$$

$$x - 2i = 0 \quad \Rightarrow \quad x = 2i$$

$$x + 2i = 0 \quad \Rightarrow \quad x = -2i.$$

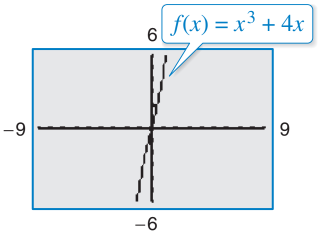


Figure 2.33

In the graph in Figure 2.33, only the real zero $x = 0$ appears as an x-intercept.

CHECKPOINT Now try Exercise 11.

Example 3 shows how to use the methods described in Sections 2.2 and 2.3 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including complex zeros.

Example 3 Finding the Zeros of a Polynomial Function

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all the zeros of f.

Solution

The possible rational zeros are $\pm1, \pm2, \pm4$, and ±8. The graph shown in Figure 2.34 indicates that 1 and -2 are likely zeros, and that 1 is possibly a repeated zero because it appears that the graph touches (but does not cross) the x-axis at this point. Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$

$$= (x - 1)(x - 1)(x + 2)(x^2 + 4).$$

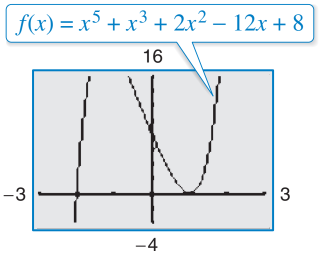


Figure 2.34

By factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right)$$

$$= (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1, x = 1, x = -2, x = 2i, \text{ and } x = -2i$$

Note from the graph of f shown in Figure 2.34 that the *real* zeros are the only ones that appear as x-intercepts.

CHECKPOINT Now try Exercise 35.

Conjugate Pairs

In Example 3, note that the two complex zeros are **conjugates.** That is, they are of the forms $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only when the polynomial function has *real coefficients*. For instance, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

Example 4 Finding a Polynomial with Given Zeros

Find a *fourth-degree* polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

CHECKPOINT Now try Exercise 47.

Example 5 Finding a Polynomial with Given Zeros

Find a *cubic* polynomial function f with real coefficients that has 2 and $1 - i$ as zeros, and $f(1) = 3$.

Solution

Because $1 - i$ is a zero of f, the conjugate $1 + i$ must also be a zero.

$$\begin{aligned} f(x) &= a(x - 2)[x - (1 - i)][x - (1 + i)] \\ &= a(x - 2)[x^2 - x(1 + i) - x(1 - i) + 1 - i^2] \\ &= a(x - 2)(x^2 - 2x + 2) \\ &= a(x^3 - 4x^2 + 6x - 4) \end{aligned}$$

To find the value of a, use the fact that $f(1) = 3$ to obtain

$$a[(1)^3 - 4(1)^2 + 6(1) - 4)] = 3.$$

Thus, $a = -3$ and you can conclude that

$$\begin{aligned} f(x) &= -3(x^3 - 4x^2 + 6x - 4) \\ &= -3x^3 + 12x^2 - 18x + 12. \end{aligned}$$

CHECKPOINT Now try Exercise 51.

Factoring a Polynomial

The Linear Factorization Theorem states that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

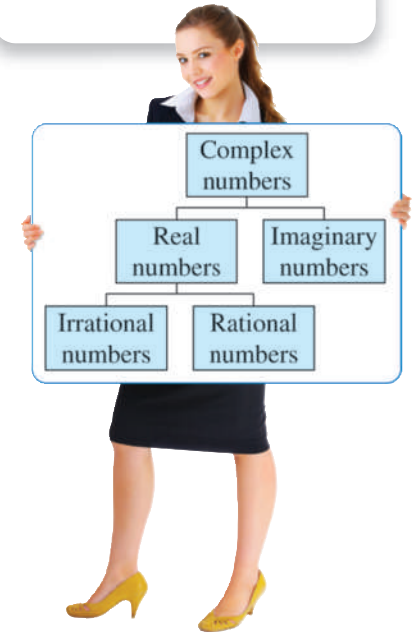
This result, however, includes the possibility that some of the values of c_i are complex. The following theorem states that even when you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

Factors of a Polynomial (See the proof on page 177.)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Study Tip

Recall that irrational and rational numbers are subsets of the set of real numbers, and the real numbers are a subset of the set of complex numbers.



A quadratic factor with no real zeros is said to be **prime** or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals*. For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals, but *reducible* over the reals.

Example 6 Factoring a Polynomial

Write the polynomial $f(x) = x^4 - x^2 - 20$

a. as the product of factors that are irreducible over the *rationals*,

b. as the product of linear factors and quadratic factors that are irreducible over the *reals*, and

c. in completely factored form.

Solution

a. Begin by factoring the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x - 2i)(x + 2i).$$

CHECKPOINT Now try Exercise 55.

In Example 6, notice from the completely factored form that the *fourth*-degree polynomial has *four* zeros.

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Throughout this chapter, the results and theorems have been stated in terms of zeros of polynomial functions. Be sure you see that the same results could have been stated in terms of solutions of polynomial equations. This is true because the zeros of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

are precisely the solutions of the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0 = 0.$$

Example 7 Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \qquad \text{and} \qquad x - (1 - 3i)$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r}
x^2 - x - 6 \\
500x^2 - 2x + 10 \overline{\big)\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2} \\
-x^3 - 4x^2 + 2x \\
\underline{-x^3 + 2x^2 - 10x} \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are

$$x = 1 + 3i,\ x = 1 - 3i,\ x = 3, \text{ and } x = -2.$$

CHECKPOINT Now try Exercise 61.

Graphical Solution

You can use a graphing utility to determine that $x = -2$ and $x = 3$ are x-intercepts of the graph of f (see Figure 2.35).

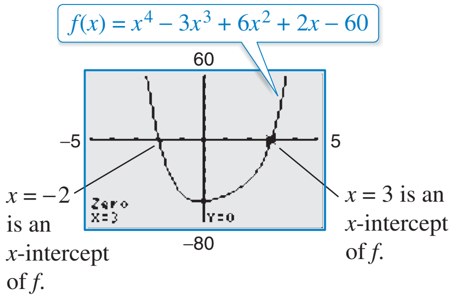


Figure 2.35

Because $1 + 3i$ is a zero of f, you know that the conjugate $1 - 3i$ must also be a zero. So, you can conclude that the zeros of f are

$$x = 1 + 3i,\ x = 1 - 3i,\ x = 3, \text{ and } x = -2.$$

In Example 7, if you were not told that $1 + 3i$ is a zero of f, you could still find all the zeros of the function by using synthetic division to find the real zeros -2 and 3. Then, you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

2.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

1. The _______ of _______ states that if $f(x)$ is a polynomial function of degree n $(n > 0)$, then f has at least one zero in the complex number system.

2. A quadratic factor that cannot be factored as a product of linear factors containing real numbers is said to be _______ over the _______ .

3. How many linear factors does a polynomial function f of degree n have, where $n > 0$?

4. Three of the zeros of a fourth-degree polynomial function f are -1, 3, and $2i$. What is the other zero of f?

Procedures and Problem Solving

Zeros of a Polynomial Function **In Exercises 5–8, match the function with its exact number of zeros.**

5. $f(x) = x^3 + x$ (a) 1 zero

6. $f(x) = -x + 7$ (b) 2 zeros

7. $f(x) = x^5 + 9x^3$ (c) 3 zeros

8. $f(x) = x^2 - 14x + 49$ (d) 5 zeros

Real and Complex Zeros of a Polynomial Function **In Exercises 9–12, confirm that the function has the indicated zeros.**

9. $f(x) = x^2 + 25;\ -5i, 5i$

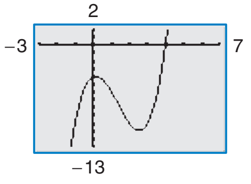
10. $f(x) = x^2 + 2;\ -\sqrt{2}i, \sqrt{2}i$

11. $f(x) = x^3 + 9x;\ 0, -3i, 3i$

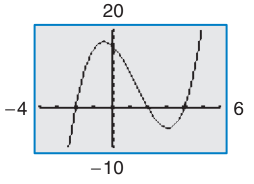
12. $f(x) = x^3 + 49x;\ 0, -7i, 7i$

Comparing the Zeros and the x-Intercepts **In Exercises 13–16, find all the zeros of the function. Is there a relationship between the number of real zeros and the number of x-intercepts of the graph? Explain.**

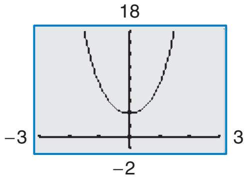
13. $f(x) = x^3 - 4x^2 + x - 4$



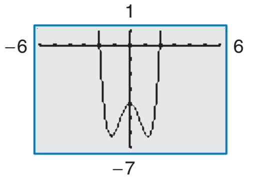
14. $f(x) = x^3 - 4x^2 - 4x + 16$



15. $f(x) = x^4 + 4x^2 + 4$



16. $f(x) = x^4 - 3x^2 - 4$



Finding the Zeros of a Polynomial Function **In Exercises 17–36, find all the zeros of the function and write the polynomial as a product of linear factors. Use a graphing utility to verify your results graphically. (If possible, use the graphing utility to verify the complex zeros.)**

17. $h(x) = x^2 - 4x + 1$

18. $g(x) = x^2 + 10x + 23$

19. $f(x) = x^2 - 12x + 26$

20. $f(x) = x^2 + 6x - 2$

21. $f(x) = x^2 + 25$

22. $f(x) = x^2 + 36$

23. $f(x) = 16x^4 - 81$

24. $f(y) = 81y^4 - 625$

25. $f(z) = z^2 - z + 56$

26. $h(x) = x^2 - 4x - 3$

27. $f(x) = x^4 + 10x^2 + 9$

28. $f(x) = x^4 + 29x^2 + 100$

29. $f(x) = 3x^3 - 5x^2 + 48x - 80$

30. $f(x) = 3x^3 - 2x^2 + 75x - 50$

31. $f(t) = t^3 - 3t^2 - 15t + 125$

32. $f(x) = x^3 + 11x^2 + 39x + 29$

33. $f(x) = 5x^3 - 9x^2 + 28x + 6$

34. $f(s) = 3s^3 - 4s^2 + 8s + 8$

35. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

36. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

Using the Zeros to Find x-Intercepts **In Exercises 37–44, (a) find all zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the x-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only x-intercepts.**

37. $f(x) = x^2 - 14x + 46$

38. $f(x) = x^2 - 12x + 34$

39. $f(x) = 2x^3 - 3x^2 + 8x - 12$

40. $f(x) = 2x^3 - 5x^2 + 18x - 45$

41. $f(x) = x^3 - 11x + 150$

42. $f(x) = x^3 + 10x^2 + 33x + 34$

43. $f(x) = x^4 + 25x^2 + 144$

44. $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

Finding a Polynomial with Given Zeros **In Exercises 45–50, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

45. $2, i, -i$ **46.** $3, 4i, -4i$

47. $2, 2, 4 - i$ **48.** $-1, -1, 2 + 5i$

49. $0, -5, 1 + \sqrt{2}i$ **50.** $0, 4, 1 + \sqrt{2}i$

Finding a Polynomial with Given Zeros **In Exercises 51–54, a polynomial function *f* with real coefficients has the given degree, zeros, and solution point. Write the function (a) in completely factored form and (b) in polynomial form.**

	Degree	*Zeros*	*Solution Point*
51.	4	$1, -2, 2i$	$f(-1) = 10$
52.	4	$-1, 2, i$	$f(1) = 8$
53.	3	$-1, 2 + \sqrt{5}i$	$f(-2) = 42$
54.	3	$-2, 2 + 2\sqrt{2}i$	$f(-1) = -34$

Factoring a Polynomial **In Exercises 55–58, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.**

55. $f(x) = x^4 - 6x^2 - 7$ **56.** $f(x) = x^4 + 6x^2 - 27$

57. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

58. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$

(*Hint:* One factor is $x^2 + 4$.)

Finding the Zeros of a Polynomial Function **In Exercises 59–66, use the given zero to find all the zeros of the function.**

	Function	*Zero*
59.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
60.	$f(x) = x^3 + x^2 + 9x + 9$	$3i$
61.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
62.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
63.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
64.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$
65.	$h(x) = 8x^3 - 14x^2 + 18x - 9$	$\frac{1}{2}(1 - \sqrt{5}i)$
66.	$f(x) = 25x^3 - 55x^2 - 54x - 18$	$\frac{1}{5}(-2 + \sqrt{2}i)$

Using a Graph to Locate the Real Zeros **In Exercises 67–70, (a) use a graphing utility to find the real zeros of the function, and then (b) use the real zeros to find the exact values of the complex zeros.**

67. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

68. $f(x) = x^3 + 4x^2 + 14x + 20$

69. $h(x) = 8x^3 - 14x^2 + 18x - 9$

70. $f(x) = 25x^3 - 55x^2 - 54x - 18$

71. ***Why you should learn it*** *(p. 135)* A football is kicked off the ground with an initial upward velocity of 48 feet per second. The football's height h (in feet) is given by

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$$h(t) = -16t^2 + 48t, \quad 0 \le t \le 3$$

where t is the time (in seconds). Does the football reach a height of 50 feet? Explain.

72. Marketing The demand equation for a microwave is $p = 140 - 0.001x$, where p is the unit price (in dollars) of the microwave and x is the number of units produced and sold. The cost equation for the microwave is $C = 40x + 150{,}000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit P obtained by producing and selling x units is given by $P = R - C = xp - C$. Is there a price p that yields a profit of \$3 million? Explain.

Conclusions

True or False? **In Exercises 73 and 74, decide whether the statement is true or false. Justify your answer.**

73. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

74. If $[x + (4 + 3i)]$ is a factor of a polynomial function f with real coefficients, then $[x - (4 + 3i)]$ is also a factor of f.

75. Writing Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

76. CAPSTONE Use a graphing utility to graph the function given by $f(x) = x^4 - 4x^2 + k$ for different values of k. Find the values of k such that the zeros of f satisfy the specified characteristics.

(a) Four real zeros

(b) Two real zeros, each of multiplicity 2

(c) Two real zeros and two complex zeros

(d) Four complex zeros

Cumulative Mixed Review

Identifying the Vertex of a Quadratic Function **In Exercises 77–80, describe the graph of the function and identify the vertex.**

77. $f(x) = x^2 - 7x - 8$ **78.** $f(x) = -x^2 + x + 6$

79. $f(x) = 6x^2 + 5x - 6$ **80.** $f(x) = 4x^2 + 2x - 12$

2.6 Rational Functions and Asymptotes

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

Example 1 Finding the Domain of a Rational Function

Find the domain of $f(x) = 1/x$ and discuss the behavior of f near any excluded x-values.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

From the table, note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. Because $f(x)$ decreases without bound from the left and increases without bound from the right, you can conclude that f is not continuous. The graph of f is shown in Figure 2.36.

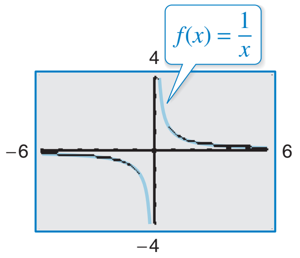


Figure 2.36

CHECKPOINT Now try Exercise 5.

Technology Tip

The graphing utility graphs in this section and the next section were created using the *dot* mode. A blue curve is placed behind the graphing utility's display to indicate where the graph should appear. You will learn more about how graphing utilities graph rational functions in the next section.

What you should learn

- Find the domains of rational functions.
- Find vertical and horizontal asymptotes of graphs of rational functions.
- Use rational functions to model and solve real-life problems.

Why you should learn it

Rational functions are convenient in modeling a wide variety of real-life problems, such as environmental scenarios. For instance, Exercise 45 on page 148 shows how to determine the cost of removing pollutants from a river.

Explore the Concept

Use the *table* and *trace* features of a graphing utility to verify that the function $f(x) = 1/x$ in Example 1 is not continuous.

Vertical and Horizontal Asymptotes

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \to -\infty \text{ as } x \to 0^-}_{f(x) \text{ decreases without bound as } x \text{ approaches 0 from the left.}} \qquad \underbrace{f(x) \to \infty \text{ as } x \to 0^+}_{f(x) \text{ increases without bound as } x \text{ approaches 0 from the right.}}$$

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 2.37. From this figure you can see that the graph of f also has a **horizontal asymptote**—the line $y = 0$. This means the values of

$$f(x) = \frac{1}{x}$$

approach zero as x increases or decreases without bound.

$$\underbrace{f(x) \to 0 \text{ as } x \to -\infty}_{f(x) \text{ approaches 0 as } x \text{ decreases without bound.}} \qquad \underbrace{f(x) \to 0 \text{ as } x \to \infty}_{f(x) \text{ approaches 0 as } x \text{ increases without bound.}}$$

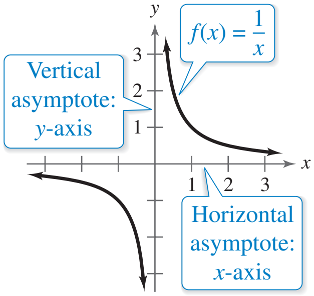


Figure 2.37

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f when

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f when

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

Explore the Concept

Use a table of values to determine whether the functions in Figure 2.38 are continuous. When the graph of a function has an asymptote, can you conclude that the function is not continuous? Explain.

Figure 2.38 shows the vertical and horizontal asymptotes of the graphs of three rational functions. Note in Figure 2.38 that eventually (as $x \to \infty$ or $x \to -\infty$) the distance between the horizontal asymptote and the points on the graph must approach zero.

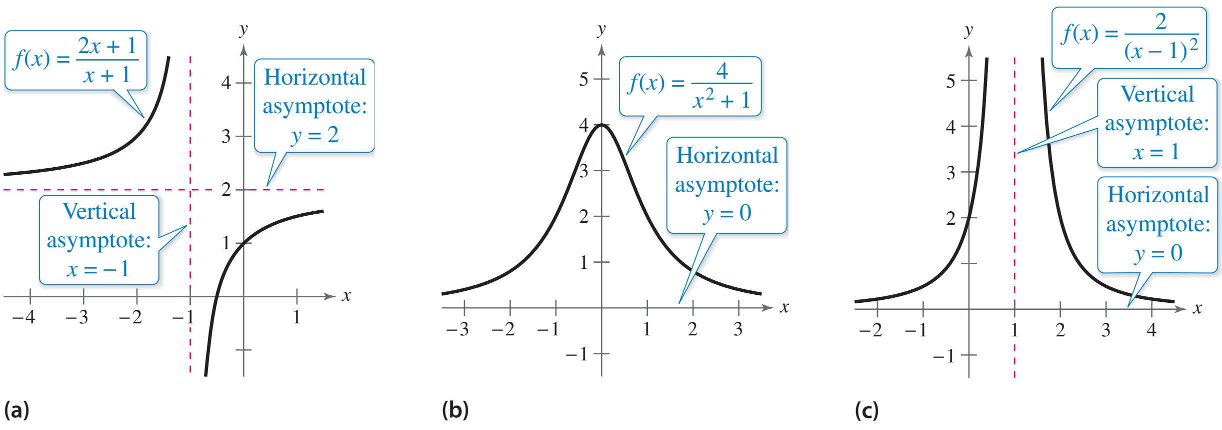


Figure 2.38

The graphs of $f(x) = 1/x$ in Figure 2.37 and $f(x) = (2x + 1)/(x + 1)$ in Figure 2.38 (a) are **hyperbolas.** You will study hyperbolas in Section 9.3.

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.

2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

a. If $n < m$, then the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

b. If $n = m$, then the graph of f has the line

$$y = \frac{a_n}{b_m}$$

as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.

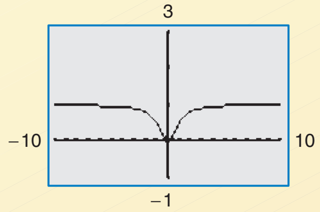
c. If $n > m$, then the graph of f has no horizontal asymptote.

What's Wrong?

You use a graphing utility to graph

$$y_1 = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}$$

as shown in the figure. You use the graph to conclude that the graph of y_1 has the line $y = 1$ as a horizontal asymptote. What's wrong?



Example 2 Finding Vertical and Horizontal Asymptotes

Find all asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

Solution

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$3x^2 + 1 = 0 \qquad \text{Set denominator equal to zero.}$$

Because this equation has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 2.39.

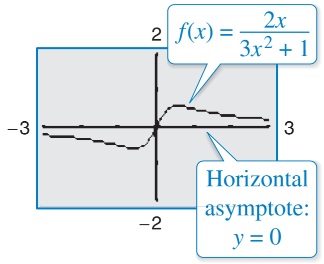


Figure 2.39

b. For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$x^2 - 1 = 0 \qquad \text{Set denominator equal to zero.}$$

$$(x + 1)(x - 1) = 0 \qquad \text{Factor.}$$

$$x + 1 = 0 \quad \Rightarrow \quad x = -1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 1 = 0 \quad \Rightarrow \quad x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes, as shown in Figure 2.40.

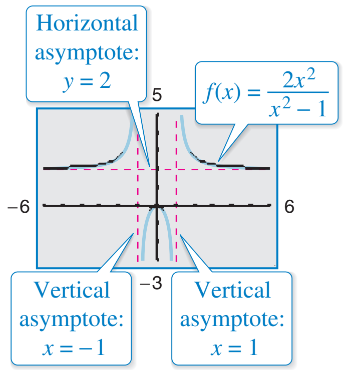


Figure 2.40

CHECKPOINT Now try Exercise 17.

Values for which a rational function is undefined (the denominator is zero) result in a vertical asymptote or a hole in the graph, as shown in Example 3.

Example 3 Finding Asymptotes and Holes

Find all asymptotes and holes in the graph of

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}.$$

Solution

For this rational function the degree of the numerator is *equal to* the degree of the denominator. The leading coefficients of the numerator and denominator are both 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote, as shown in Figure 2.41. To find any holes in the graph, note that the function is undefined at $x = -2$ and $x = 3$. Because $x = -2$ is not a vertical asymptote of the function, there is a hole in the graph at $x = -2$. To find the y-coordinate of the hole, substitute $x = -2$ into the simplified form of the function.

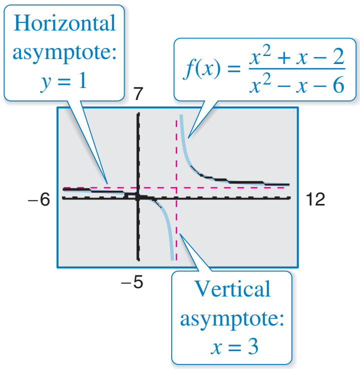


Figure 2.41

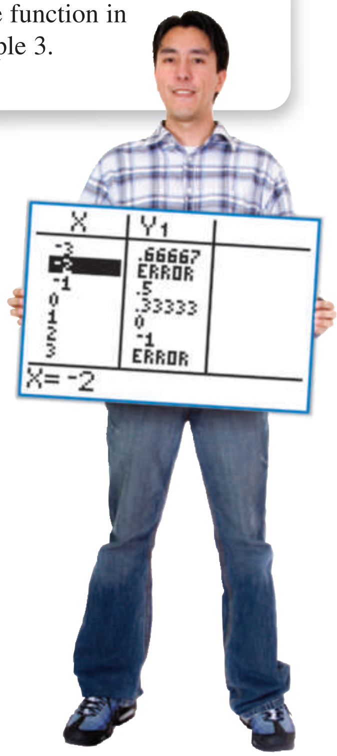
$$y = \frac{x - 1}{x - 3} = \frac{-2 - 1}{-2 - 3} = \frac{3}{5}$$

So, the graph of the rational function has a hole at $\left(-2, \frac{3}{5}\right)$.

CHECKPOINT Now try Exercise 23.

Technology Tip

Graphing utilities are limited in their resolution and therefore may not show a break or hole in the graph. You can use the *table* feature of a graphing utility to verify the values of x at which a function is not defined. Try doing this for the function in Example 3.



Example 4 Finding a Function's Domain and Asymptotes

For the function f, find (a) the domain of f, (b) the vertical asymptote of f, and (c) the horizontal asymptote of f.

$$f(x) = \frac{3x^3 + 7x^2 + 2}{-2x^3 + 16}$$

Solution

a. Because the denominator is zero when $-2x^3 + 16 = 0$, solve this equation to determine that the domain of f is all real numbers except $x = 2$.

b. Because the denominator of f has a zero at $x = 2$, and 2 is not a zero of the numerator, the graph of f has the vertical asymptote $x = 2$.

c. Because the degrees of the numerator and denominator are the same, and the leading coefficient of the numerator is 3 and the leading coefficient of the denominator is -2, the horizontal asymptote of f is $y = -\frac{3}{2}$.

CHECKPOINT Now try Exercise 25.

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Application

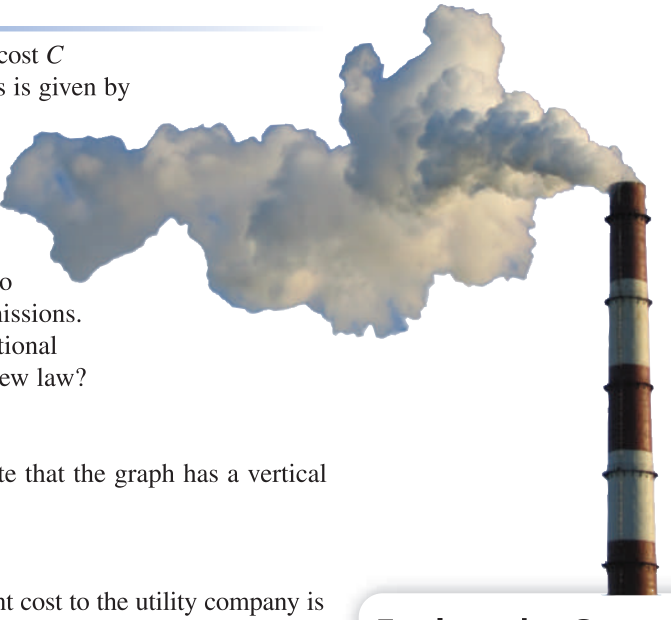
There are many examples of asymptotic behavior in real life. For instance, Example 5 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

Example 5 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing $p\%$ of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}$$

for $0 \le p < 100$. Use a graphing utility to graph this function. You are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?



Solution

The graph of this function is shown in Figure 2.42. Note that the graph has a vertical asymptote at

$$p = 100.$$

Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000p}{100 - p} \qquad \text{Write original function.}$$

$$C = \frac{80{,}000(85)}{100 - 85} \qquad \text{Substitute 85 for } p.$$

$$\approx \$453{,}333. \qquad \text{Simplify.}$$

If the new law increases the percent removal to 90%, the cost will be

$$C = \frac{80{,}000p}{100 - p} \qquad \text{Write original function.}$$

$$C = \frac{80{,}000(90)}{100 - 90} \qquad \text{Substitute 90 for } p.$$

$$= \$720{,}000. \qquad \text{Simplify.}$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \text{Subtract 85\% removal cost from 90\% removal cost.}$$

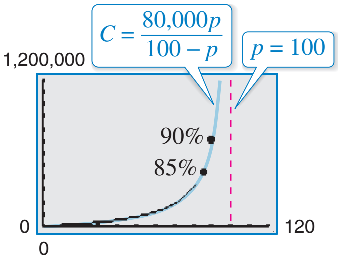


Figure 2.42

CHECKPOINT Now try Exercise 45.

Explore the Concept

The *table* feature of a graphing utility can be used to estimate vertical and horizontal asymptotes of rational functions. Use the *table* feature to find any vertical or horizontal asymptotes of

$$f(x) = \frac{2x}{x + 1}.$$

Write a statement explaining how you found the asymptote(s) using the table.

2.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called ________ .

2. If $f(x) \to \pm\infty$ as $x \to a$ from the left (or right), then $x = a$ is a ________ of the graph of f.

3. What feature of the graph of $y = \dfrac{9}{x - 3}$ can you find by solving $x - 3 = 0$?

4. Is $y = \dfrac{2}{3}$ a horizontal asymptote of the function $y = \dfrac{2x}{3x^2 - 5}$?

Procedures and Problem Solving

Finding the Domain of a Rational Function **In Exercises 5–10, (a) find the domain of the function, (b) complete each table, and (c) discuss the behavior of f near any excluded x-values.**

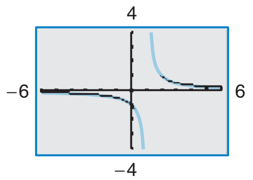
x	$f(x)$
0.5	
0.9	
0.99	
0.999	

x	$f(x)$
1.5	
1.1	
1.01	
1.001	

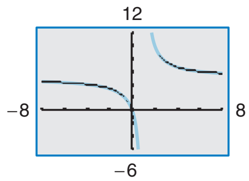
x	$f(x)$
5	
10	
100	
1000	

x	$f(x)$
-5	
-10	
-100	
-1000	

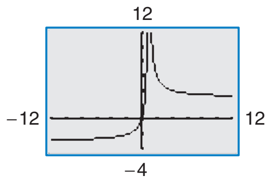
5. $f(x) = \dfrac{1}{x - 1}$



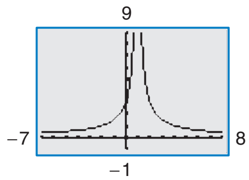
6. $f(x) = \dfrac{5x}{x - 1}$



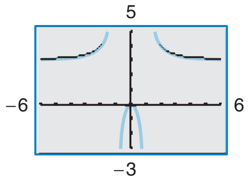
7. $f(x) = \dfrac{3x}{|x - 1|}$



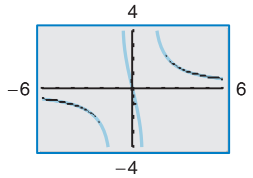
8. $f(x) = \dfrac{3}{|x - 1|}$



9. $f(x) = \dfrac{3x^2}{x^2 - 1}$

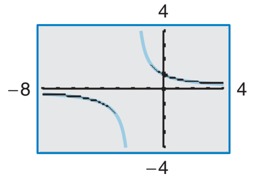


10. $f(x) = \dfrac{4x}{x^2 - 1}$

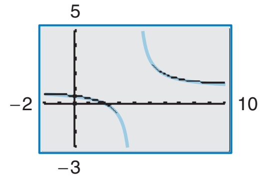


Identifying Graphs of Rational Functions **In Exercises 11–16, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

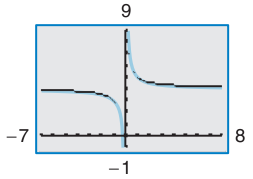
(a)



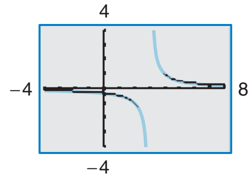
(b)



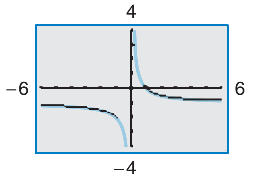
(c)



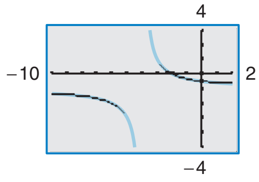
(d)



(e)



(f)



11. $f(x) = \dfrac{2}{x + 2}$

12. $f(x) = \dfrac{1}{x - 3}$

13. $f(x) = \dfrac{4x + 1}{x}$

14. $f(x) = \dfrac{1 - x}{x}$

15. $f(x) = \dfrac{x - 2}{x - 4}$

16. $f(x) = -\dfrac{x + 2}{x + 4}$

Finding Vertical and Horizontal Asymptotes **In Exercises 17–20, find any asymptotes of the graph of the rational function. Verify your answers by using a graphing utility to graph the function.**

17. $f(x) = \dfrac{1}{x^2}$

18. $f(x) = \dfrac{3}{(x-2)^3}$

19. $f(x) = \dfrac{2x^2}{x^2 + x - 6}$

20. $f(x) = \dfrac{x^2 - 4x}{x^2 - 4}$

Finding Asymptotes and Holes **In Exercises 21–24, find any asymptotes and holes in the graph of the rational function. Verify your answers by using a graphing utility.**

21. $f(x) = \dfrac{x(2+x)}{2x - x^2}$

22. $f(x) = \dfrac{x^2 + 2x + 1}{2x^2 - x - 3}$

23. $f(x) = \dfrac{x^2 - 25}{x^2 + 5x}$

24. $f(x) = \dfrac{3 - 14x - 5x^2}{3 + 7x + 2x^2}$

Finding a Function's Domain and Asymptotes **In Exercises 25–28, (a) find the domain of the function, (b) decide whether the function is continuous, and (c) identify any horizontal and vertical asymptotes. Verify your answer to part (a) both graphically by using a graphing utility and numerically by creating a table of values.**

25. $f(x) = \dfrac{3x^2 + x - 5}{x^2 + 1}$

26. $f(x) = \dfrac{3x^2 + 1}{x^2 + x + 9}$

27. $f(x) = \dfrac{x^2 + 3x - 4}{-x^3 + 27}$

28. $f(x) = \dfrac{4x^3 - x^2 + 3}{3x^3 + 24}$

Algebraic-Graphical-Numerical **In Exercises 29–32, (a) determine the domains of f and g, (b) find any vertical asymptotes and holes in the graphs of f and g, (c) compare f and g by completing the table, (d) use a graphing utility to graph f and g, and (e) explain why the differences in the domains of f and g are not shown in their graphs.**

29. $f(x) = \dfrac{x^2 - 16}{x - 4}, \quad g(x) = x + 4$

x	1	2	3	4	5	6	7
$f(x)$							
$g(x)$							

30. $f(x) = \dfrac{x^2 - 9}{x - 3}, \quad g(x) = x + 3$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

31. $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}, \quad g(x) = \dfrac{x - 1}{x - 3}$

x	-2	-1	0	1	2	3	4
$f(x)$							
$g(x)$							

32. $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}, \quad g(x) = \dfrac{x + 2}{x - 1}$

x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							

Exploration **In Exercises 33–36, determine the value that the function f approaches as the magnitude of x increases. Is $f(x)$ greater than or less than this value when x is positive and large in magnitude? What about when x is negative and large in magnitude?**

33. $f(x) = 4 - \dfrac{1}{x}$

34. $f(x) = 2 + \dfrac{1}{x - 3}$

35. $f(x) = \dfrac{2x - 1}{x - 3}$

36. $f(x) = \dfrac{2x - 1}{x^2 + 1}$

Finding the Zeros of a Rational Function **In Exercises 37–44, find the zeros (if any) of the rational function. Use a graphing utility to verify your answer.**

37. $g(x) = \dfrac{x^2 - 4}{x + 3}$

38. $g(x) = \dfrac{x^3 - 8}{x^2 + 4}$

39. $f(x) = 1 - \dfrac{2}{x - 5}$

40. $h(x) = 5 + \dfrac{3}{x^2 + 1}$

41. $g(x) = \dfrac{x^2 - 2x - 3}{x^2 + 1}$

42. $g(x) = \dfrac{x^2 - 5x + 6}{x^2 + 4}$

43. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - 7x + 3}$

44. $f(x) = \dfrac{2x^2 + 3x - 2}{x^2 + x - 2}$

45. ***Why you should learn it*** *(p. 142)* The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by

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$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

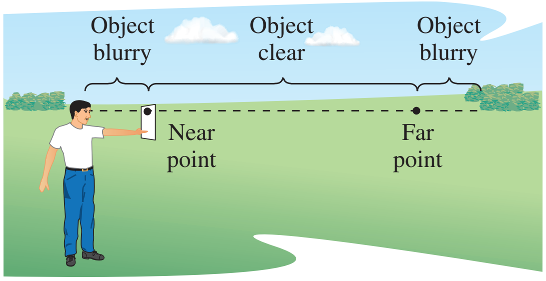
(a) Find the costs of removing 10%, 40%, and 75% of the pollutants.

(b) Use a graphing utility to graph the cost function. Explain why you chose the values that you used in your viewing window.

(c) According to this model, is it possible to remove 100% of the pollutants? Explain.

46. MODELING DATA

The endpoints of the interval over which distinct vision is possible are called the *near point* and *far point* of the eye (see figure). With increasing age these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years).



Age, x	Near point, y
16	3.0
32	4.7
44	9.8
50	19.7
60	39.4

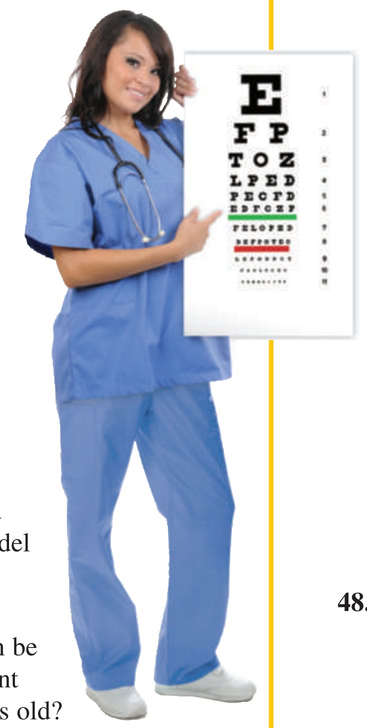
(a) Find a rational model for the data. Take the reciprocals of the near points to generate the points $(x, 1/y)$. Use the *regression* feature of a graphing utility to find a linear model for the data. The resulting line has the form

$$\frac{1}{y} = ax + b.$$

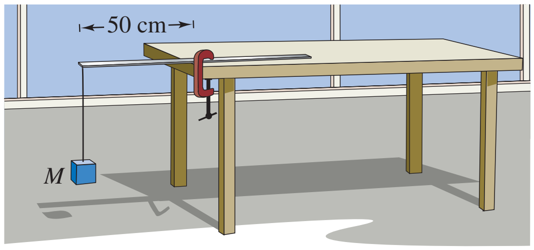
Solve for y.

(b) Use the *table* feature of the graphing utility to create a table showing the predicted near point based on the model for each of the ages in the original table.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.



47. Physics Consider a physics laboratory experiment designed to determine an unknown mass. A flexible metal meter stick is clamped to a table with 50 centimeters overhanging the edge (see figure). Known masses M ranging from 200 grams to 2000 grams are attached to the end of the meter stick. For each mass, the meter stick is displaced vertically and then allowed to oscillate. The average time t (in seconds) of one oscillation for each mass is recorded in the table.



Mass, M	Time, t
200	0.450
400	0.597
600	0.712
800	0.831
1000	0.906
1200	1.003
1400	1.088
1600	1.126
1800	1.218
2000	1.338

A model for the data is given by

$$t = \frac{38M + 16{,}965}{10(M + 5000)}.$$

(a) Use the *table* feature of a graphing utility to create a table showing the estimated time based on the model for each of the masses shown in the table. What can you conclude?

(b) Use the model to approximate the mass of an object when the average time for one oscillation is 1.056 seconds.

48. Biology The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is given by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \geq 0$$

where t is the time in years.

(a) Use a graphing utility to graph the model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases? Explain.

49. Military Science The table shows the numbers N (in thousands) of Department of Defense personnel from 1990 through 2008. The data can be modeled by

$$N = \frac{77.095t^2 - 216.04t + 2050}{0.052t^2 - 0.08t + 1}, \quad 0 \le t \le 18$$

where t is the year, with $t = 0$ corresponding to 1990. (Source: U.S. Department of Defense)

Year	Number, N (in thousands)
1990	2044
1991	1986
1992	1807
1993	1705
1994	1610
1995	1518
1996	1472
1997	1439
1998	1407
1999	1386
2000	1384
2001	1385
2002	1414
2003	1434
2004	1427
2005	1389
2006	1385
2007	1380
2008	1402

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model represent the data?

(b) Use the model to estimate the numbers of Department of Defense personnel in the years 2009, 2010, and 2011. Are the estimates reasonable?

(c) Find the horizontal asymptote of the graph of the model. Does it seem realistic that the numbers of Department of Defense personnel will keep getting closer to the value of the asymptote? Explain.

Conclusions

True or False? **In Exercises 50 and 51, determine whether the statement is true or false. Justify your answer.**

50. A rational function can have infinitely many vertical asymptotes.

51. A rational function must have at least one vertical asymptote.

52. Think About It Describe the possible features of the graph of a rational function f at $x = c$, when c is not in the domain of f.

53. Think About It A real zero of the numerator of a rational function f is $x = c$. Must $x = c$ also be a zero of f? Explain.

54. Think About It When the graph of a rational function f has a vertical asymptote at $x = 4$, can f have a common factor of $(x - 4)$ in the numerator and denominator? Explain.

55. Exploration Use a graphing utility to compare the graphs of y_1 and y_2.

$$y_1 = \frac{3x^3 - 5x^2 + 4x - 5}{2x^2 - 6x + 7}, \quad y_2 = \frac{3x^3}{2x^2}$$

Start with a viewing window of $-5 \le x \le 5$ and $-10 \le y \le 10$, and then zoom out. Make a conjecture about how the graph of a rational function f is related to the graph of $y = a_nx^n/b_mx^m$, where a_nx^n is the leading term of the numerator of f and b_mx^m is the leading term of the denominator of f.

56. CAPSTONE Write a rational function f that has the specified characteristics. (There are many correct answers.)

(a) Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 0$
Zero: $x = 1$

(b) Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 0$
Zero: $x = 2$

(c) Vertical asymptotes: $x = -2, x = 1$
Horizontal asymptote: $y = 2$
Zeros: $x = 3, x = -3$

(d) Vertical asymptotes: $x = -1, x = 2$
Horizontal asymptote: $y = -2$
Zeros: $x = -2, x = 3$

Cumulative Mixed Review

Finding the Equation of a Line **In Exercises 57–60, write the general form of the equation of the line that passes through the points.**

57. $(3, 2), (0, -1)$

58. $(-6, 1), (4, -5)$

59. $(2, 7), (3, 10)$

60. $(0, 0), (-9, 4)$

Long Division of Polynomials **In Exercises 61–64, divide using long division.**

61. $(x^2 + 5x + 6) \div (x - 4)$

62. $(x^2 - 10x + 15) \div (x - 3)$

63. $(2x^4 + x^2 - 11) \div (x^2 + 5)$

64. $(4x^5 + 3x^3 - 10) \div (2x + 3)$

2.7 Graphs of Rational Functions

The Graph of a Rational Function

To sketch the graph of a rational function, use the following guidelines.

Guidelines for Graphing Rational Functions

Let

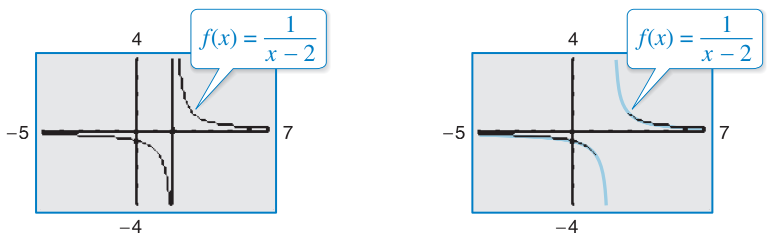
$$f(x) = N(x)/D(x)$$

where $N(x)$ and $D(x)$ are polynomials.

1. Simplify f, if possible. Any restrictions on the domain of f not in the simplified function should be listed.
2. Find and plot the y-intercept (if any) by evaluating $f(0)$.
3. Find the zeros of the numerator (if any) by setting the numerator equal to zero. Then plot the corresponding x-intercepts.
4. Find the zeros of the denominator (if any) by setting the denominator equal to zero. Then sketch the corresponding vertical asymptotes using dashed vertical lines and plot the corresponding holes using open circles.
5. Find and sketch any other asymptotes of the graph using dashed lines.
6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes, excluding any points where f is not defined.

Technology Tip

Some graphing utilities have difficulty graphing rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. Notice that the graph in Figure 2.43(a) should consist of two *unconnected* portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the *mode* of the graphing utility to *dot mode*. The problem with this mode is that the graph is then represented as a collection of dots rather than as a smooth curve, as shown in Figure 2.43(b). In this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear. [See Figure 2.43(b).]



(a) Connected mode **(b) Dot mode**

Figure 2.43

What you should learn

- Analyze and sketch graphs of rational functions.
- Sketch graphs of rational functions that have slant asymptotes.
- Use graphs of rational functions to model and solve real-life problems.

Why you should learn it

The graph of a rational function provides a good indication of the behavior of a mathematical model. Exercise 89 on page 159 models the concentration of a chemical in the bloodstream after injection.

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Library of Parent Functions: Rational Function

The simplest type of rational function is the *parent rational function* $f(x) = 1/x$, also known as the *reciprocal function*. The basic characteristics of the parent rational function are summarized below and on the inside cover of this text.

Graph of $f(x) = \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

No intercepts

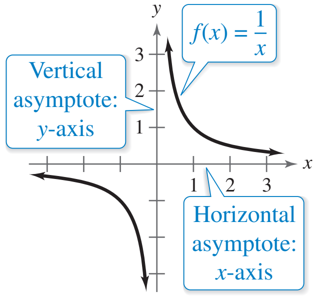
Decreasing on $(-\infty, 0)$ and $(0, \infty)$

Odd function

Origin symmetry

Vertical asymptote: y-axis

Horizontal asymptote: x-axis



Explore the Concept

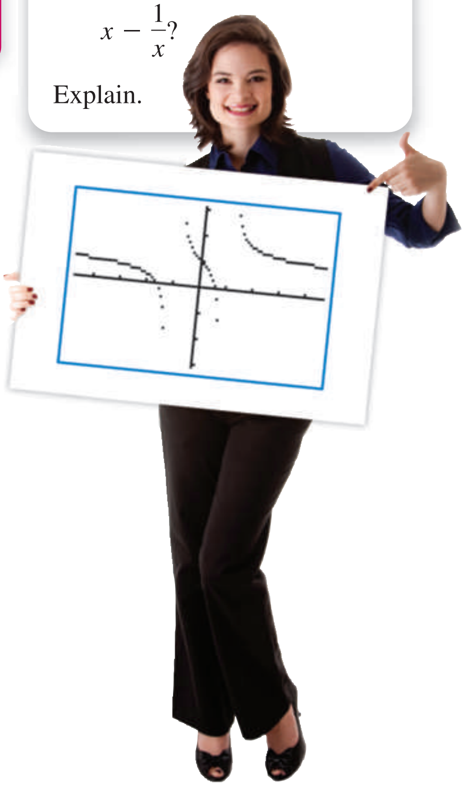
Use a graphing utility to graph

$$f(x) = 1 + \frac{1}{x - \frac{1}{x}}.$$

Set the graphing utility to *dot* mode and use a decimal viewing window. Use the *trace* feature to find three "holes" or "breaks" in the graph. Do all three holes or breaks represent zeros of the denominator

$$x - \frac{1}{x}?$$

Explain.



Example 1 Library of Parent Functions: $f(x) = 1/x$

Sketch the graph of the function and describe how the graph is related to the graph of $f(x) = 1/x$.

a. $g(x) = \dfrac{-1}{x + 2}$

b. $h(x) = \dfrac{1}{x - 1} + 3$

Solution

a. With respect to the graph of $f(x) = 1/x$, the graph of g is obtained by a *reflection* in the y-axis and a horizontal shift two units *to the left*, as shown in Figure 2.44. Confirm this with a graphing utility.

b. With respect to the graph of $f(x) = 1/x$, the graph of h is obtained by a horizontal shift one unit *to the right* and a vertical shift three units *upward*, as shown in Figure 2.45. Confirm this with a graphing utility.

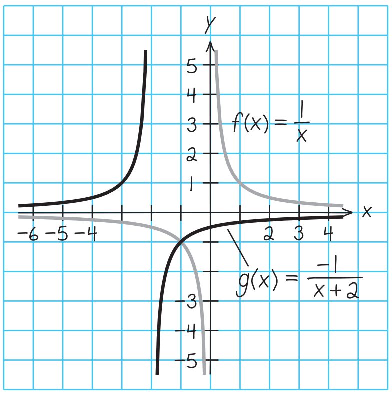


Figure 2.44

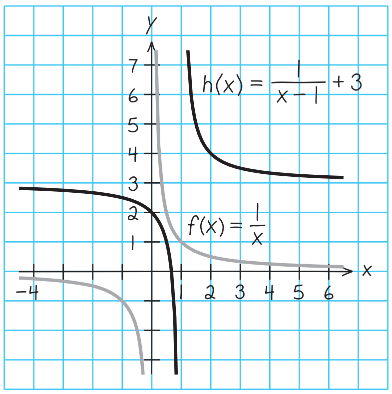


Figure 2.45

CHECKPOINT Now try Exercise 5.

Example 2 Sketching the Graph of a Rational Function

Sketch the graph of

$$g(x) = \frac{3}{x - 2}$$

by hand.

Solution

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: None, because $3 \neq 0$

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-4	1	2	3	5
$g(x)$	-0.5	-3	Undefined	3	1

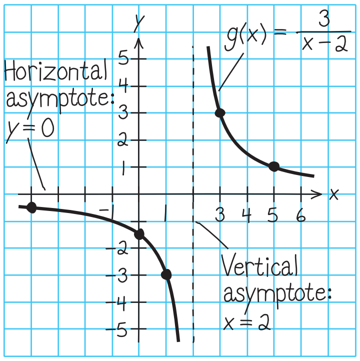


Figure 2.46

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.46. Confirm this with a graphing utility.

CHECKPOINT Now try Exercise 17.

Note that the graph of g in Example 2 is a vertical stretch and a right shift of the graph of

$$f(x) = \frac{1}{x}$$

because

$$g(x) = \frac{3}{x - 2} = 3\left(\frac{1}{x - 2}\right) = 3f(x - 2).$$

Study Tip

Note in Examples 2–6 that the vertical asymptotes are included in the tables of additional points. This is done to emphasize numerically the behavior of the graph of the function.

Example 3 Sketching the Graph of a Rational Function

Sketch the graph of

$$f(x) = \frac{2x - 1}{x}$$

by hand.

Solution

y-intercept: None, because $x = 0$ is not in the domain

x-intercept: $\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$ when $x = \frac{1}{2}$

Vertical asymptote: $x = 0$, zero of denominator

Horizontal asymptote: $y = 2$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-4	-1	0	$\frac{1}{4}$	4
$f(x)$	2.25	3	Undefined	-2	1.75

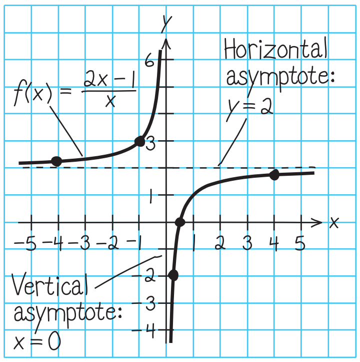


Figure 2.47

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.47. Confirm this with a graphing utility.

CHECKPOINT Now try Exercise 21.

Example 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$.

Solution

Factor the denominator to determine more easily the zeros of the denominator.

$$f(x) = \frac{x}{x^2 - x - 2} = \frac{x}{(x+1)(x-2)}$$

y-intercept: $(0, 0)$, because $f(0) = 0$

x-intercept: $(0, 0)$

Vertical asymptotes: $x = -1$, $x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-3	-1	-0.5	1	2	3
$f(x)$	-0.3	Undefined	0.4	-0.5	Undefined	0.75

The graph is shown in Figure 2.48.

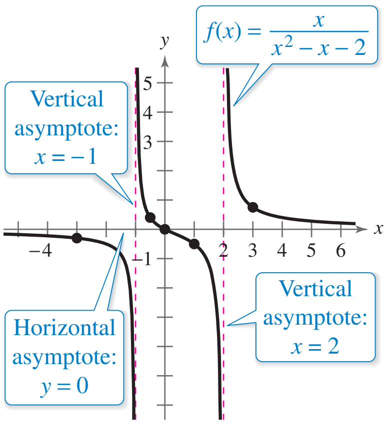


Figure 2.48

CHECKPOINT Now try Exercise 27.

Example 5 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2 - 9}{x^2 - 2x - 3}$.

Solution

By factoring the numerator and denominator, you have

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+1)} = \frac{x+3}{x+1}, \quad x \neq 3.$$

y-intercept: $(0, 3)$, because $f(0) = 3$

x-intercept: $(-3, 0)$, because $x + 3 = 0$ when $x = -3$

Vertical asymptote: $x = -1$, zero of (simplified) denominator

Hole: $\left(3, \frac{3}{2}\right)$, f is not defined at $x = 3$

Horizontal asymptote: $y = 1$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-5	-2	-1	-0.5	1	3	4
$f(x)$	0.5	-1	Undefined	5	2	Undefined	1.4

The graph is shown in Figure 2.49.

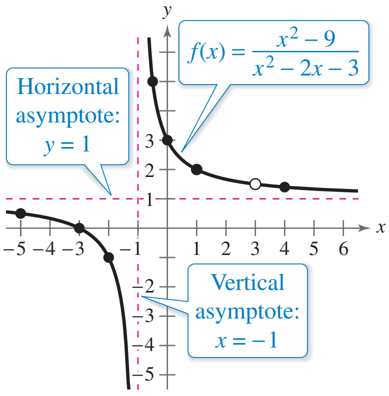


Figure 2.49 *Hole at $x = 3$*

CHECKPOINT Now try Exercise 29.

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 2.50. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\substack{\text{Slant asymptote} \\ (y = x - 2)}} + \frac{2}{x + 1}.$$

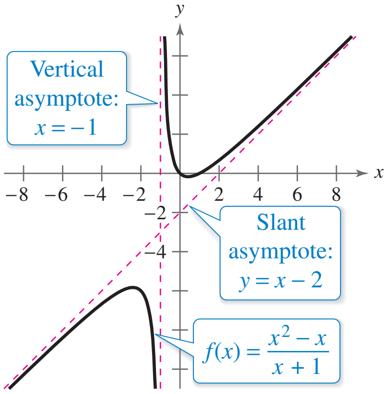


Figure 2.50

As x increases or decreases without bound, the remainder term

$$\frac{2}{x + 1}$$

approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 2.50.

Explore the Concept

Do you think it is possible for the graph of a rational function to cross its horizontal asymptote or its slant asymptote? Use the graphs of the following functions to investigate this question. Write a summary of your conclusion. Explain your reasoning.

$$f(x) = \frac{x}{x^2 + 1}$$

$$g(x) = \frac{2x}{3x^2 - 2x + 1}$$

$$h(x) = \frac{x^3}{x^2 + 1}$$

Example 6 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

First write $f(x)$ in two different ways. Factoring the numerator

$$f(x) = \frac{x^2 - x - 2}{x - 1} = \frac{(x - 2)(x + 1)}{x - 1}$$

enables you to recognize the x-intercepts. Long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

enables you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-intercept: $(0, 2)$, because $f(0) = 2$

x-intercepts: $(-1, 0)$ and $(2, 0)$

Vertical asymptote: $x = 1$, zero of denominator

Horizontal asymptote: None, because degree of $N(x) >$ degree of $D(x)$

Slant asymptote: $y = x$

Additional points:

x	-2	0.5	1	1.5	3
$f(x)$	-1.33	4.5	Undefined	-2.5	2

The graph is shown in Figure 2.51.

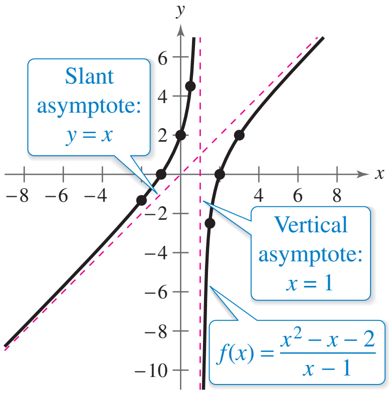


Figure 2.51

CHECKPOINT Now try Exercise 51.

Application

Example 7 Publishing

A rectangular page is designed to contain 48 square inches of print. The margins on each side of the page are $1\frac{1}{2}$ inches wide. The margins at the top and bottom are each 1 inch deep. What should the dimensions of the page be so that the minimum amount of paper is used?

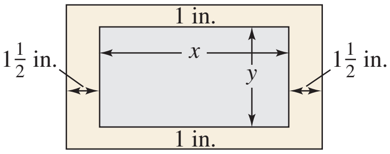


Figure 2.52

Graphical Solution

Let A be the area to be minimized. From Figure 2.52, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown in Figure 2.53. Because x represents the width of the printed area, you need consider only the portion of the graph for which x is positive. Using the *minimum* feature of a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches by } y + 2 \approx 7.6 \text{ inches.}$$

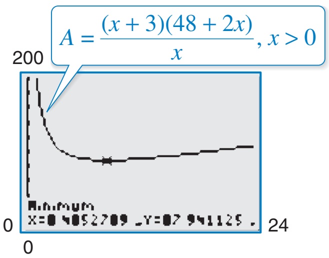


Figure 2.53

CHECKPOINT Now try Exercise 85.

Numerical Solution

Let A be the area to be minimized. From Figure 2.52, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function

$$y_1 = \frac{(x + 3)(48 + 2x)}{x}$$

beginning at $x = 1$. From the table, you can see that the minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 2.54. To approximate the minimum value of y_1 to one decimal place, change the table to begin at $x = 8$ and set the table step to 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 2.55. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches by } y + 2 \approx 7.6 \text{ inches.}$$

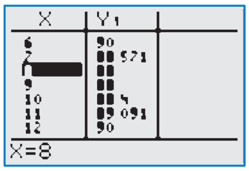


Figure 2.54

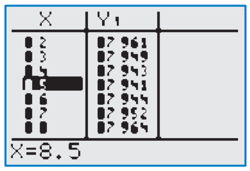


Figure 2.55

If you go on to take a course in calculus, you will learn an analytic technique for finding the exact value of x that produces a minimum area in Example 7. In this case, that value is

$$x = 6\sqrt{2} \approx 8.485.$$

2.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. For the rational function $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a _______ (or oblique) _______ .

2. The graph of $f(x) = 1/x$ has a _______ asymptote at $x = 0$.

3. Does the graph of $f(x) = \dfrac{x^3 - 1}{x^2 + 2}$ have a slant asymptote?

4. Using long division, you find that $f(x) = \dfrac{x^2 + 1}{x + 1} = x - 1 + \dfrac{2}{x + 1}$. What is the slant asymptote of the graph of f?

Procedures and Problem Solving

Library of Parent Functions: $f(x) = 1/x$ **In Exercises 5–8, sketch the graph of the function g and describe how the graph is related to the graph of $f(x) = 1/x$.**

5. $g(x) = \dfrac{1}{x - 4}$ **6.** $g(x) = \dfrac{-1}{x} - 5$

7. $g(x) = \dfrac{-1}{x + 3}$ **8.** $g(x) = \dfrac{1}{x + 6} - 2$

Describing a Transformation of $f(x) = 2/x$ **In Exercises 9–12, use a graphing utility to graph $f(x) = 2/x$ and the function g in the same viewing window. Describe the relationship between the two graphs.**

9. $g(x) = f(x) + 1$ **10.** $g(x) = f(x - 1)$

11. $g(x) = -f(x)$ **12.** $g(x) = \frac{1}{2}f(x + 2)$

Describing a Transformation of $f(x) = 2/x^2$ **In Exercises 13–16, use a graphing utility to graph $f(x) = 2/x^2$ and the function g in the same viewing window. Describe the relationship between the two graphs.**

13. $g(x) = f(x) - 2$ **14.** $g(x) = -f(x)$

15. $g(x) = f(x - 2)$ **16.** $g(x) = \frac{1}{4}f(x)$

Sketching the Graph of a Rational Function **In Exercises 17–32, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and holes. Use a graphing utility to verify your graph.**

17. $f(x) = \dfrac{1}{x + 2}$ **18.** $f(x) = \dfrac{1}{x - 6}$

19. $C(x) = \dfrac{5 + 2x}{1 + x}$ **20.** $P(x) = \dfrac{1 - 3x}{1 - x}$

21. $f(t) = \dfrac{1 - 2t}{t}$ **22.** $g(x) = \dfrac{1}{x + 2} + 2$

23. $f(x) = \dfrac{x^2}{x^2 - 4}$ **24.** $g(x) = \dfrac{x}{x^2 - 9}$

25. $g(x) = \dfrac{4(x + 1)}{x(x - 4)}$ **26.** $h(x) = \dfrac{2}{x^2(x - 3)}$

27. $f(x) = \dfrac{3x}{x^2 - x - 2}$ **28.** $f(x) = \dfrac{2x}{x^2 + x - 2}$

29. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6}$ **30.** $g(x) = \dfrac{5(x + 4)}{x^2 + x - 12}$

31. $f(x) = \dfrac{x^2 - 1}{x + 1}$ **32.** $f(x) = \dfrac{x^2 - 16}{x - 4}$

Finding the Domain and Asymptotes **In Exercises 33–42, use a graphing utility to graph the function. Determine its domain and identify any vertical or horizontal asymptotes.**

33. $f(x) = \dfrac{2 + x}{1 - x}$ **34.** $f(x) = \dfrac{3 - x}{2 - x}$

35. $f(t) = \dfrac{3t + 1}{t}$ **36.** $h(x) = \dfrac{x - 2}{x - 3}$

37. $h(t) = \dfrac{4}{t^2 + 1}$ **38.** $g(x) = -\dfrac{x}{(x - 2)^2}$

39. $f(x) = \dfrac{x + 1}{x^2 - x - 6}$ **40.** $f(x) = \dfrac{x + 4}{x^2 + x - 6}$

41. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$ **42.** $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

Exploration **In Exercises 43–48, use a graphing utility to graph the function. What do you observe about its asymptotes?**

43. $h(x) = \dfrac{6x}{\sqrt{x^2 + 1}}$ **44.** $f(x) = -\dfrac{x}{\sqrt{9 + x^2}}$

45. $g(x) = \dfrac{4|x - 2|}{x + 1}$ **46.** $f(x) = -\dfrac{8|3 + x|}{x - 2}$

47. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$ **48.** $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

A Rational Function with a Slant Asymptote **In Exercises 49–56, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, and slant asymptotes.**

49. $f(x) = \dfrac{2x^2 + 1}{x}$ **50.** $g(x) = \dfrac{1 - x^2}{x}$

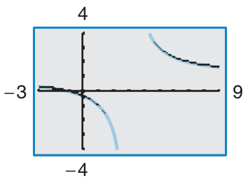
51. $h(x) = \dfrac{x^2}{x - 1}$ **52.** $f(x) = \dfrac{x^3}{x^2 - 1}$

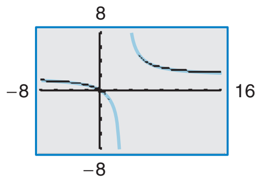
53. $g(x) = \dfrac{x^3}{2x^2 - 8}$ **54.** $f(x) = \dfrac{x^3}{x^2 + 4}$

55. $f(x) = \dfrac{x^3 + 2x^2 + 4}{2x^2 + 1}$ **56.** $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

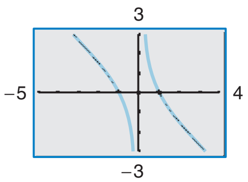
Finding the *x*-Intercepts **In Exercises 57–60, use the graph to estimate any *x*-intercepts of the rational function. Set $y = 0$ and solve the resulting equation to confirm your result.**

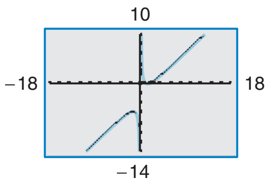
57. $y = \dfrac{x + 1}{x - 3}$ **58.** $y = \dfrac{2x}{x - 3}$





59. $y = \dfrac{1}{x} - x$ **60.** $y = x - 3 + \dfrac{2}{x}$





Finding the Domain and Asymptotes **In Exercises 61–64, use a graphing utility to graph the rational function. Determine the domain of the function and identify any asymptotes.**

61. $y = \dfrac{2x^2 + x}{x + 1}$ **62.** $y = \dfrac{x^2 + 5x + 8}{x + 3}$

63. $y = \dfrac{1 + 3x^2 - x^3}{x^2}$ **64.** $y = \dfrac{12 - 2x - x^2}{2(4 + x)}$

Finding Asymptotes and Holes **In Exercises 65–70, find all vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes in the graph of the function. Then use a graphing utility to verify your result.**

65. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 4}$ **66.** $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$

67. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - x - 6}$ **68.** $f(x) = \dfrac{3x^2 - 8x + 4}{2x^2 - 3x - 2}$

69. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

70. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

Finding *x*-Intercepts Graphically **In Exercises 71–82, use a graphing utility to graph the function and determine any *x*-intercepts. Set $y = 0$ and solve the resulting equation to confirm your result.**

71. $y = \dfrac{1}{x + 5} + \dfrac{4}{x}$ **72.** $y = \dfrac{2}{x + 1} - \dfrac{3}{x}$

73. $y = \dfrac{1}{x + 2} + \dfrac{2}{x + 4}$ **74.** $y = \dfrac{2}{x + 2} - \dfrac{3}{x - 1}$

75. $y = x - \dfrac{6}{x - 1}$ **76.** $y = x - \dfrac{9}{x}$

77. $y = x + 2 - \dfrac{1}{x + 1}$

78. $y = 2x - 1 + \dfrac{1}{x - 2}$

79. $y = x + 1 + \dfrac{2}{x - 1}$

80. $y = x + 2 + \dfrac{2}{x + 2}$

81. $y = x + 3 - \dfrac{2}{2x - 1}$

82. $y = x - 1 - \dfrac{2}{2x - 3}$

83. Chemistry A 1000-liter tank contains 50 liters of a 25% brine solution. You add x liters of a 75% brine solution to the tank.

(a) Show that the concentration C (the ratio of brine to the total solution) of the final mixture is given by

$$C = \frac{3x + 50}{4(x + 50)}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the function. As the tank is filled, what happens to the rate at which the concentration of brine increases? What percent does the concentration of brine appear to approach?

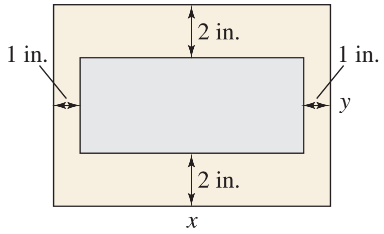
84. Geometry A rectangular region of length x and width y has an area of 500 square meters.

(a) Write the width y as a function of x.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Sketch a graph of the function and determine the width of the rectangle when $x = 30$ meters.

85. Publishing A page that is x inches wide and y inches high contains 30 square inches of print. The margins at the top and bottom are 2 inches deep and the margins on each side are 1 inch wide (see figure).



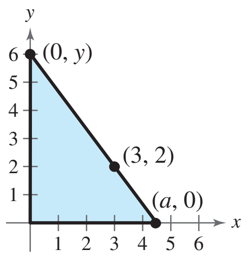
(a) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 11)}{x - 2}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used. Verify your answer numerically using the *table* feature of the graphing utility.

86. Geometry A right triangle is formed in the first quadrant by the x-axis, the y-axis, and a line segment through the point $(3, 2)$ (see figure).



(a) Show that an equation of the line segment is given by

$$y = \frac{2(a - x)}{a - 3}, \quad 0 \le x \le a.$$

(b) Show that the area of the triangle is given by

$$A = \frac{a^2}{a - 3}.$$

(c) Use a graphing utility to graph the area function and estimate the value of a that yields a minimum area. Estimate the minimum area. Verify your answer numerically using the *table* feature of the graphing utility.

87. Cost Management The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \ge 1$$

where x is the order size (in hundreds). Use a graphing utility to graph the cost function. From the graph, estimate the order size that minimizes cost.

88. Cost Management The cost C of producing x units of a product is given by $C = 0.2x^2 + 10x + 5$, and the average cost per unit is given by

$$\overline{C} = \frac{C}{x} = \frac{0.2x^2 + 10x + 5}{x}, \quad x > 0.$$

Sketch the graph of the average cost function, and estimate the number of units that should be produced to minimize the average cost per unit.

89. *Why you should learn it* *(p. 151)* The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t \ge 0.$$

(a) Determine the horizontal asymptote of the function and interpret its meaning in the context of the problem.

(b) Use a graphing utility to graph the function and approximate the time when the bloodstream concentration is greatest.

(c) Use the graphing utility to determine when the concentration is less than 0.345.

90. Algebraic-Graphical-Numerical A driver averaged 50 miles per hour on the round trip between Baltimore, Maryland and Philadelphia, Pennsylvania, 100 miles away. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = \dfrac{25x}{x - 25}$.

(b) Determine the vertical and horizontal asymptotes of the function.

(c) Use a graphing utility to complete the table. What do you observe?

x	30	35	40	45	50	55	60
y							

(d) Use the graphing utility to graph the function.

(e) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

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91. MODELING DATA

Data are recorded at 124 monitoring sites throughout the United States to study national trends in air quality. The table shows the mean amount A of carbon monoxide (in parts per million) recorded at these sites in each year from 1999 through 2008. (Source: Environmental Protection Agency)

Year	Amount, A (in parts per million)
1999	3.9
2000	3.5
2001	3.3
2002	2.9
2003	2.7
2004	2.5
2005	2.3
2006	2.2
2007	2.0
2008	1.9

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let $t = 9$ represent 1999. Use the graphing utility to plot the data and graph the model in the same viewing window.

(b) Find a rational model for the data. Take the reciprocal of A to generate the points $(t, 1/A)$. Use the *regression* feature of the graphing utility to find a linear model for these data. The resulting line has the form $1/A = at + b$. Solve for A. Use the graphing utility to plot the data and graph the rational model in the same viewing window.

(c) Use the *table* feature of the graphing utility to create a table showing the mean amounts of carbon monoxide generated by each model for the years in the original table. Which model do you prefer? Why?

92. Biology A herd of elk is released onto state game lands. The expected population P of the herd can be modeled by the equation $P = (10 + 2.7t)/(1 + 0.1t)$, where t is the time in years since the initial number of elk were released.

(a) State the domain of the model. Explain your answer.

(b) Find the initial number of elk in the herd.

(c) Find the populations of elk after 25, 50, and 100 years.

(d) Is there a limit to the size of the herd? If so, what is the expected population?

Use a graphing utility to confirm your results for parts (a) through (d).

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Conclusions

True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. The graph of a rational function is continuous only when the denominator is a constant polynomial.

94. The graph of a rational function can never cross one of its asymptotes.

Think About It **In Exercises 95 and 96, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function might indicate that there should be one.**

95. $h(x) = \dfrac{6 - 2x}{3 - x}$

96. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

97. Writing Write a set of guidelines for finding all the asymptotes of a rational function given that the degree of the numerator is not more than 1 greater than the degree of the denominator.

98. CAPSTONE Write a rational function that has the specified characteristics. (There are many correct answers.)

(a) Vertical asymptote: $x = -2$

Slant asymptote: $y = x + 1$

Zero of the function: $x = 2$

(b) Vertical asymptote: $x = -4$

Slant asymptote: $y = x - 2$

Zero of the function: $x = 3$

Cumulative Mixed Review

Simplifying Exponential Expressions **In Exercises 99–102, simplify the expression.**

99. $\left(\dfrac{x}{8}\right)^{-3}$

100. $(4x^2)^{-2}$

101. $\dfrac{3^{7/6}}{3^{1/6}}$

102. $\dfrac{(x^{-2})(x^{1/2})}{(x^{-1})(x^{5/2})}$

Finding the Domain and Range of a Function **In Exercises 103–106, use a graphing utility to graph the function and find its domain and range.**

103. $f(x) = \sqrt{6 + x^2}$

104. $f(x) = \sqrt{121 - x^2}$

105. $f(x) = -|x + 9|$

106. $f(x) = -x^2 + 9$

107. *Make a Decision* To work an extended application analyzing the median sales prices of existing one-family homes, visit this textbook's *Companion Website*. (Data Source: National Association of Realtors)

2.8 Quadratic Models

What you should learn

- Classify scatter plots.
- Use scatter plots and a graphing utility to find quadratic models for data.
- Choose a model that best fits a set of data.

Why you should learn it

Many real-life situations can be modeled by quadratic equations. For instance, in Exercise 17 on page 165, a quadratic equation is used to model the monthly precipitation for San Francisco, California.

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Classifying Scatter Plots

In real life, many relationships between two variables are parabolic, as in Section 2.1, Example 5. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

Example 1 Classifying Scatter Plots

Decide whether each set of data could be better modeled by a linear model,

$$y = ax + b$$

a quadratic model,

$$y = ax^2 + bx + c$$

or neither.

a. (0.9, 1.7), (1.2, 2.0), (1.3, 1.9), (1.4, 2.1), (1.6, 2.5), (1.8, 2.8), (2.1, 3.0), (2.5, 3.4), (2.9, 3.7), (3.2, 3.9), (3.3, 4.1), (3.6, 4.4), (4.0, 4.7), (4.2, 4.8), (4.3, 5.0)

b. (0.9, 3.2), (1.2, 4.0), (1.3, 4.1), (1.4, 4.4), (1.6, 5.1), (1.8, 6.0), (2.1, 7.6), (2.5, 9.8), (2.9, 12.4), (3.2, 14.3), (3.3, 15.2), (3.6, 18.1), (4.0, 22.7), (4.2, 24.9), (4.3, 27.2)

c. (0.9, 1.2), (1.2, 6.5), (1.3, 9.3), (1.4, 11.6), (1.6, 15.2), (1.8, 16.9), (2.1, 14.7), (2.5, 8.1), (2.9, 3.7), (3.2, 5.8), (3.3, 7.1), (3.6, 11.5), (4.0, 20.2), (4.2, 23.7), (4.3, 26.9)

Solution

a. Begin by entering the data into a graphing utility. Then display the scatter plot, as shown in Figure 2.56. From the scatter plot, it appears the data follow a linear pattern. So, the data can be better modeled by a linear function.

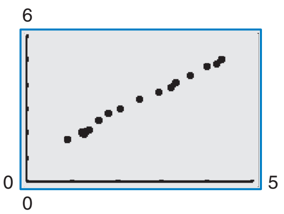

Figure 2.56

b. Enter the data into a graphing utility and then display the scatter plot (see Figure 2.57). From the scatter plot, it appears the data follow a parabolic pattern. So, the data can be better modeled by a quadratic function.

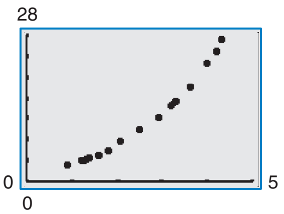

Figure 2.57

c. Enter the data into a graphing utility and then display the scatter plot (see Figure 2.58). From the scatter plot, it appears the data do not follow either a linear or a parabolic pattern. So, the data cannot be modeled by either a linear function or a quadratic function.

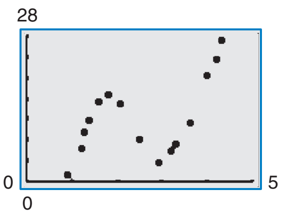

Figure 2.58

CHECKPOINT Now try Exercise 5.

Fitting a Quadratic Model to Data

In Section 1.7, you created scatter plots of data and used a graphing utility to find the least squares regression lines for the data. You can use a similar procedure to find a model for nonlinear data. Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

Example 2 Fitting a Quadratic Model to Data

A study was done to compare the speed x (in miles per hour) with the mileage y (in miles per gallon) of an automobile. The results are shown in the table.

a. Use a graphing utility to create a scatter plot of the data.

b. Use the *regression* feature of the graphing utility to find a model that best fits the data.

c. Approximate the speed at which the mileage is the greatest.

Speed, x	Mileage, y
15	22.3
20	25.5
25	27.5
30	29.0
35	28.7
40	29.9
45	30.4
50	30.2
55	30.0
60	28.8
65	27.4
70	25.3
75	23.3

Solution

a. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.59. From the scatter plot, you can see that the data appear to follow a parabolic pattern.

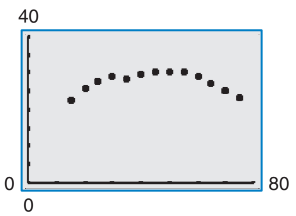

Figure 2.59

b. Using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 2.60. So, the quadratic equation that best fits the data is given by

$$y = -0.0082x^2 + 0.75x + 13.5. \quad \text{Quadratic model}$$

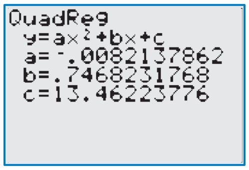

Figure 2.60

c. Graph the data and the model in the same viewing window, as shown in Figure 2.61. Use the *maximum* feature or the *zoom* and *trace* features of the graphing utility to approximate the speed at which the mileage is greatest. You should obtain a maximum of approximately (46, 31), as shown in Figure 2.61. So, the speed at which the mileage is greatest is about 46 miles per hour.

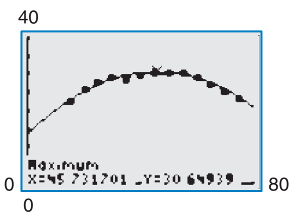

Figure 2.61

CHECKPOINT Now try Exercise 17.

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Example 3 Fitting a Quadratic Model to Data

A basketball is dropped from a height of about 5.25 feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second. The results are shown in the table. Use a graphing utility to find a model that best fits the data. Then use the model to predict the time when the basketball will hit the ground.

Time, x	Height, y
0.0	5.23594
0.02	5.20353
0.04	5.16031
0.06	5.09910
0.08	5.02707
0.099996	4.95146
0.119996	4.85062
0.139992	4.74979
0.159988	4.63096
0.179988	4.50132
0.199984	4.35728
0.219984	4.19523
0.23998	4.02958
0.25993	3.84593
0.27998	3.65507
0.299976	3.44981
0.319972	3.23375
0.339961	3.01048
0.359961	2.76921
0.379951	2.52074
0.399941	2.25786
0.419941	1.98058
0.439941	1.63488

Solution

Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.62.

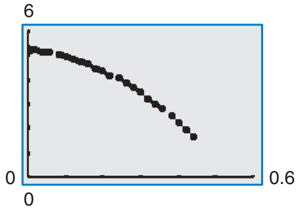


Figure 2.62

From the scatter plot, you can see that the data show a parabolic trend. So, using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 2.63. The quadratic model that best fits the data is given by $y = -15.449x^2 - 1.30x + 5.2$.

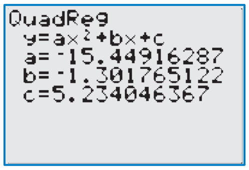


Figure 2.63

You can graph the data and the model in the same viewing window to see that the model fits the data well, as shown in Figure 2.64.

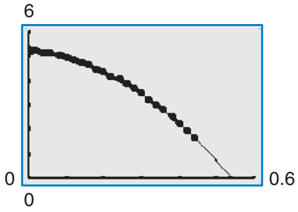


Figure 2.64

Using this model, you can predict the time when the basketball will hit the ground by substituting 0 for y and solving the resulting equation for x.

$$y = -15.449x^2 - 1.30x + 5.2 \qquad \text{Write original model.}$$

$$0 = -15.449x^2 - 1.30x + 5.2 \qquad \text{Substitute 0 for } y.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-1.30) \pm \sqrt{(-1.30)^2 - 4(-15.449)(5.2)}}{2(-15.449)} \qquad \text{Substitute for } a, b, \text{ and } c.$$

$$\approx 0.54 \qquad \text{Choose positive solution.}$$

So, the solution is about 0.54 second. In other words, the basketball will continue to fall for about $0.54 - 0.44 = 0.1$ second more before hitting the ground.

CHECKPOINT Now try Exercise 19.

Choosing a Model

Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data, using the *Library of Parent Functions*, and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.

Example 4 Choosing a Model

The table shows the amounts y (in gallons per person) of regular soft drinks consumed in the United States in the years 2000 through 2007. Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data. Determine which model better fits the data. (Source: United States Department of Agriculture)

Year	Amounts, y
2000	39.4
2001	39.0
2002	38.5
2003	37.5
2004	37.0
2005	36.3
2006	35.4
2007	33.9

Solution

Let x represent the year, with $x = 0$ corresponding to 2000. Begin by entering the data into a graphing utility. Using the *regression* feature of the graphing utility, a linear model for the data is

$$y = -0.76x + 39.8 \qquad \text{Linear model}$$

and a quadratic model for the data is

$$y = -0.056x^2 - 0.37x + 39.4. \qquad \text{Quadratic model}$$

Plot the data and the linear model in the same viewing window, as shown in Figure 2.65. Then plot the data and the quadratic model in the same viewing window, as shown in Figure 2.66. To determine which model fits the data better, compare the y-values given by each model with the actual y-values. The model whose y-values are closest to the actual values is the better fit. In this case, the better-fitting model is the quadratic model.

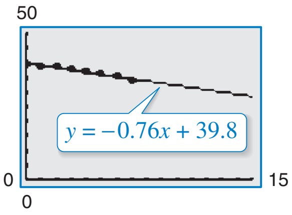


Figure 2.65

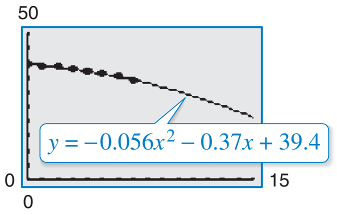


Figure 2.66

CHECKPOINT Now try Exercise 21.

Technology Tip

When you use the regression feature of a graphing utility, the program may output an "r^2-value." This r^2-value is the **coefficient of determination** of the data and gives a measure of how well the model fits the data. The coefficient of determination for the linear model in Example 4 is

$$r^2 \approx 0.97105$$

and the coefficient of determination for the quadratic model is

$$r^2 \approx 0.99226.$$

Because the coefficient of determination for the quadratic model is closer to 1, the quadratic model better fits the data.

2.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

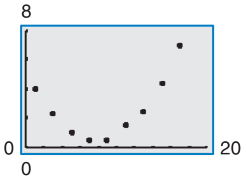
1. What type of model best represents data that follow a parabolic pattern?

2. Which coefficient of determination indicates a better model for a set of data, $r^2 = 0.0365$, or $r^2 = 0.9688$?

Procedures and Problem Solving

Classifying Scatter Plots **In Exercises 3–8, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.**

3.



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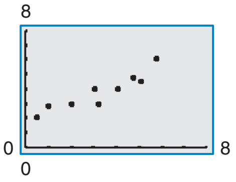
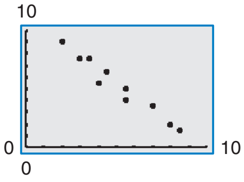


5.



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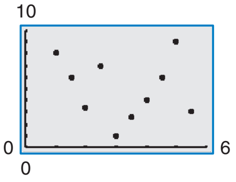

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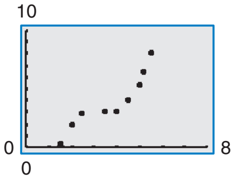

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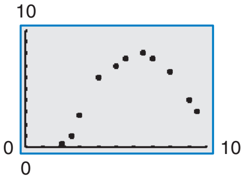


Using a Scatter Plot **In Exercises 9–16, (a) use a graphing utility to create a scatter plot of the data, (b) determine whether the data could be better modeled by a linear model or a quadratic model, (c) use the *regression* feature of the graphing utility to find a model for the data, (d) use the graphing utility to graph the model with the scatter plot from part (a), and (e) create a table comparing the original data with the data given by the model.**

9. (0, 2.1), (1, 2.4), (2, 2.5), (3, 2.8), (4, 2.9), (5, 3.0), (6, 3.0), (7, 3.2), (8, 3.4), (9, 3.5), (10, 3.6)

10. (−2, 11.0), (−1, 10.7), (0, 10.4), (1, 10.3), (2, 10.1), (3, 9.9), (4, 9.6), (5, 9.4), (6, 9.4), (7, 9.2), (8, 9.0)

11. (0, 3480), (5, 2235), (10, 1250), (15, 565), (20, 150), (25, 12), (30, 145), (35, 575), (40, 1275), (45, 2225), (50, 3500), (55, 5010)

12. (0, 6140), (2, 6815), (4, 7335), (6, 7710), (8, 7915), (10, 7590), (12, 7975), (14, 7700), (16, 7325), (18, 6820), (20, 6125), (22, 5325)

13. (1, 4.0), (2, 6.5), (3, 8.8), (4, 10.6), (5, 13.9), (6, 15.0), (7, 17.5), (8, 20.1), (9, 24.0), (10, 27.1)

14. (−6, 10.7), (−4, 9.0), (−2, 7.0), (0, 5.4), (2, 3.5), (4, 1.7), (6, −0.1), (8, −1.8), (10, −3.6), (12, −5.3)

15. (0, 587), (5, 551), (10, 512), (15, 478), (20, 436), (25, 430) (30, 424), (35, 420), (40, 423), (45, 429), (50, 444)

16. (2, 34.3), (3, 33.8), (4, 32.6), (5, 30.1), (6, 27.8), (7, 22.5), (8, 19.1), (9, 14.8), (10, 9.4). (11, 3.7), (12, −1.6)

17. ***Why you should learn it*** *(p. 161)* The table shows the monthly normal precipitation P (in inches) for San Francisco, California. (Source: U.S. National Oceanic and Atmospheric Administration)

Month	Precipitation, P
January	4.45
February	4.01
March	3.26
April	1.17
May	0.38
June	0.11
July	0.03
August	0.07
September	0.20
October	1.40
November	2.49
December	2.89

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the month, with $t = 1$ corresponding to January.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) Use the graph from part (c) to determine in which month the normal precipitation in San Francisco is the least.

LeggNet/iStockphoto.com

18. MODELING DATA

The table shows the annual sales S (in billions of dollars) of department stores in the United States from 2003 through 2008. (Source: U.S. Census Bureau)

Year	Sales, S (in billions of dollars)
2003	221.0
2004	222.0
2005	220.7
2006	219.0
2007	215.9
2008	206.1

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) Use the model to estimate the first year when the annual sales of department stores will be below \$180 billion. Is this a good model for predicting future sales? Explain.

✓ **19. MODELING DATA**

The table shows the percents P of the U.S. population who used the Internet from 2003 through 2008. (Source: U.S. Census Bureau)

Year	Percent, P
2003	55.58
2004	63.00
2005	66.33
2006	69.83
2007	71.83
2008	74.00

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) According to the model, in what year will the percent of the U.S. population who use the Internet fall below 60%? Is this a good model for making future predictions? Explain.

20. MODELING DATA

The table shows the average numbers of hours H that adults in the United States spent reading newspapers each year from 2002 through 2007. (Source: Veronis Suhler Stevenson)

Year	Hours, H
2002	188
2003	195
2004	192
2005	187
2006	178
2007	171

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) A cubic model for the data is

$$H = 0.500t^3 - 8.43t^2 + 38.9t + 140$$

which has an r^2-value of 0.9965. Use the graphing utility to graph the model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data? Explain.

(e) Which model is a better fit for the data? Explain.

(f) The projected average numbers of hours H^* that adults spent reading newspapers each year from 2008 through 2010 are shown in the table. Use the models from parts (b) and (c) to *predict* the average numbers of hours for 2008 through 2010. Explain why your values may differ from those in the table.

Year	2008	2009	2010
H^*	164	159	155

grafica 2010/used under license from Shutterstock.com

21. MODELING DATA

The table shows the numbers of televisions T (in millions) in homes in the United States from 1997 through 2008. (Source: The Nielsen Company)

Year	Televisions, T (in millions)
1997	229
1998	235
1999	240
2000	245
2001	248
2002	254
2003	260
2004	268
2005	287
2006	301
2007	311
2008	310

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data and identify the coefficient of determination.

(c) Use the graphing utility to graph the linear model with the scatter plot from part (a).

(d) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(e) Use the graphing utility to graph the quadratic model with the scatter plot from part (a).

(f) Which model is a better fit for the data? Explain.

(g) Use each model to approximate the year when the number of televisions in homes will reach 350 million.

Conclusions

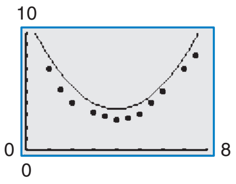
True or False? **In Exercises 22–24, determine whether the statement is true or false. Justify your answer.**

22. The graph of a quadratic model with a negative leading coefficient will have a maximum value at its vertex.

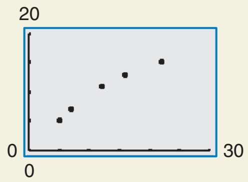
23. The graph of a quadratic model with a positive leading coefficient will have a minimum value at its vertex.

24. Data that are positively correlated are always better modeled by a linear equation than by a quadratic equation.

25. **Writing** Explain why the parabola shown in the figure is not a good fit for the data.



26. CAPSTONE The r^2-values representing the coefficients of determination for the least squares linear model and the least squares quadratic model for the data shown are given below. Which is which? Explain your reasoning.



$r^2 \approx 0.9995$

$r^2 \approx 0.9782$

Cumulative Mixed Review

Compositions of Functions **In Exercises 27–30, find (a) $f \circ g$ and (b) $g \circ f$.**

27. $f(x) = 2x - 1, \quad g(x) = x^2 + 3$

28. $f(x) = 5x + 8, \quad g(x) = 2x^2 - 1$

29. $f(x) = x^3 - 1, \quad g(x) = \sqrt[3]{x + 1}$

30. $f(x) = \sqrt[3]{x + 5}, \quad g(x) = x^3 - 5$

Testing for One-to-One Functions **In Exercises 31–34, determine algebraically whether the function is one-to-one. If it is, find its inverse function. Verify your answer graphically.**

31. $f(x) = 2x + 5$

32. $f(x) = \dfrac{x - 4}{5}$

33. $f(x) = x^2 + 5, x \geq 0$

34. $f(x) = 2x^2 - 3, x \geq 0$

Multiplying Complex Conjugates **In Exercises 35–38, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.**

35. $1 - 3i$

36. $-2 + 4i$

37. $-5i$

38. $8i$

2 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
2.1	**Analyze graphs of quadratic functions *(p. 90)*.**	Let a, b, and c be real numbers with $a \neq 0$. The function $f(x) = ax^2 + bx + c$ is called a quadratic function. Its graph is a "U-shaped" curve called a parabola.	1–6
	Write quadratic functions in standard form and use the results to sketch graphs of functions *(p. 93)*.	The quadratic function $f(x) = a(x - h)^2 + k$, $a \neq 0$, is in standard form. The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). The parabola opens upward when $a > 0$ and opens downward when $a < 0$.	7–12
	Find minimum and maximum values of quadratic functions in real-life applications *(p. 95)*.	Consider $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. If $a > 0$, then f has a *minimum* at $x = -b/(2a)$. If $a < 0$, then f has a *maximum* at $x = -b/(2a)$.	13, 14
2.2	**Use transformations to sketch graphs of polynomial functions *(p. 100)*.**	The graph of a polynomial function is continuous (no breaks, holes, or gaps) and has only smooth, rounded turns.	15–20
	Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions *(p. 102)*.	Consider $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$. ***n* is odd:** If $a_n > 0$, then the graph falls to the left and rises to the right. If $a_n < 0$, then the graph rises to the left and falls to the right. ***n* is even:** If $a_n > 0$, then the graph rises to the left and right. If $a_n < 0$, then the graph falls to the left and right.	21–26
	Find and use zeros of polynomial functions as sketching aids *(p. 104)*.	If f is a polynomial function and a is a real number, the following are equivalent: (1) $x = a$ is a *zero* of the function f, (2) $x = a$ is a *solution* of the polynomial equation $f(x) = 0$, (3) $(x - a)$ is a *factor* of the polynomial $f(x)$, and (4) $(a, 0)$ is an *x-intercept* of the graph of f.	27–38
	Use the Intermediate Value Theorem to help locate zeros of polynomial functions *(p. 108)*.	Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.	39–42
2.3	**Use long division to divide polynomials by other polynomials *(p. 113)*.**	Dividend, Quotient, Remainder, Divisor, Divisor $\frac{x^2 + 3x + 5}{x + 1} = \underbrace{x + 2}_{\text{Quotient}} + \frac{3}{x + 1}$	43–50
	Use synthetic division to divide polynomials by binomials of the form $(x - k)$ *(p. 116)*.	Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$ $\begin{array}{r\|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$ Remainder: 1 Quotient: $x^3 - 3x^2 - x + 1$	51–56
	Use the Remainder Theorem and the Factor Theorem *(p. 117)*.	**The Remainder Theorem:** If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$. **The Factor Theorem:** A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.	57–62
	Use the Rational Zero Test to determine possible rational zeros of polynomial functions *(p. 119)*.	The Rational Zero Test relates the possible rational zeros of a polynomial to the leading coefficient and to the constant term of the polynomial.	63, 64

	What did you learn?	Explanation and Examples	Review Exercises
2.3	**Use Descartes's Rule of Signs *(p. 121)* and the Upper and Lower Bound Rules *(p. 122)* to find zeros of polynomials.**	Example 9 shows how to use Descartes's Rule of Signs. Example 10 uses Descartes's Rule of Signs and the Upper and Lower Bound Rules.	65–72
2.4	**Use the imaginary unit i to write complex numbers *(p. 128)*.**	The imaginary unit i is defined as $i = \sqrt{-1}$. If a and b are real numbers, then the number $a + bi$ is a complex number, and it is written in standard form.	73–76
	Add, subtract, and multiply complex numbers *(p. 129)*.	**Sum:** $(a + bi) + (c + di) = (a + c) + (b + d)i$ **Difference:** $(a + bi) - (c + di) = (a - c) + (b - d)i$	77–88
	Use complex conjugates to write the quotient of two complex numbers in standard form *(p. 131)*.	Complex numbers of the forms $a + bi$ and $a - bi$ are complex conjugates. To write $(a + bi)/(c + di)$ in standard form, multiply by $(c - di)/(c - di)$.	89–92
	Find complex solutions of quadratic equations *(p. 132)*.	If a is a positive number, then the principal square root of the negative number $-a$ is defined as $\sqrt{-a} = \sqrt{a}i$.	93–98
2.5	**Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function *(p. 135)*.**	**The Fundamental Theorem of Algebra** If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.	99–102
	Find all zeros of polynomial functions, including complex zeros *(p. 136)*, and find conjugate pairs of complex zeros *(p. 137)*.	**Complex Zeros Occur in Conjugate Pairs** Let $f(x)$ be a polynomial function that has real coefficients. If $a + bi$ $(b \neq 0)$ is a zero of the function, the conjugate $a - bi$ is also a zero of the function.	103–120
	Find zeros of polynomials by factoring *(p. 138)*.	Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.	121–124
2.6	**Find the domains *(p. 142)* and vertical and horizontal asymptotes *(p. 143)* of rational functions.**	The domain of a rational function of x includes all real numbers except x-values that make the denominator zero.	125–136
	Use rational functions to model and solve real-life problems *(p. 146)*.	A rational function can be used to model the cost of removing a given percent of smokestack pollutants at a utility company that burns coal. (See Example 5.)	137, 138
2.7	**Analyze and sketch graphs of rational functions *(p. 151)*, including functions with slant asymptotes *(p. 155)*.**	Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a slant asymptote.	139–152
	Use rational functions to model and solve real-life problems *(p. 156)*.	A rational function can be used to model the area of a page. The model can be used to determine the dimensions of the page that use the minimum amount of paper. (See Example 7.)	153, 154
2.8	**Classify scatter plots *(p. 161)*, find quadratic models for data *(p. 162)*, and choose a model that best fits a set of data *(p. 164)*.**	Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.	155–160

2 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

2.1

Library of Parent Functions **In Exercises 1–6, sketch the graph of each function and describe how the graph is related to the graph of $y = x^2$.**

1. $y = x^2 - 2$ **2.** $y = x^2 + 4$

3. $y = (x - 2)^2$ **4.** $y = -(x + 4)^2$

5. $y = (x + 5)^2 - 2$ **6.** $y = -(x - 4)^2 + 1$

Identifying the Vertex of a Quadratic Function **In Exercises 7–10, describe the graph of the function and identify the vertex. Then, sketch the graph of the function. Identify any x-intercepts.**

7. $f(x) = \left(x + \frac{3}{2}\right)^2 + 1$ **8.** $f(x) = (x - 4)^2 - 4$

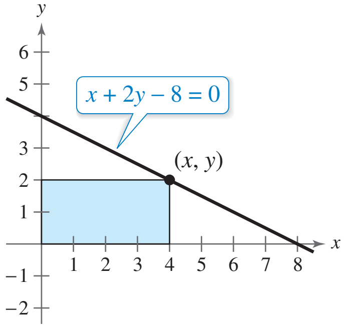
9. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$ **10.** $f(x) = 3x^2 - 12x + 11$

Writing the Equation of a Parabola in Standard Form **In Exercises 11 and 12, write the standard form of the quadratic function that has the indicated vertex and whose graph passes through the given point. Use a graphing utility to verify your result.**

11. Vertex: $(1, -4)$; Point: $(2, -3)$

12. Vertex: $(2, 3)$; Point: $(0, 2)$

13. Geometry A rectangle is inscribed in the region bounded by the x-axis, the y-axis, and the graph of $x + 2y - 8 = 0$, as shown in the figure.

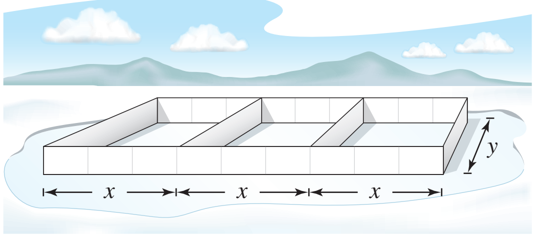


(a) Write the area A of the rectangle as a function of x. Determine the domain of the function in the context of the problem.

(b) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce a maximum area.

(c) Write the area function in standard form to find algebraically the dimensions that will produce a maximum area. Compare your results with your answer from part (b).

14. Physical Education A college has 1500 feet of portable rink boards to form three adjacent outdoor ice rinks, as shown in the figure. Determine the dimensions that will produce the maximum total area of ice surface.



2.2

Library of Parent Functions **In Exercises 15–20, sketch the graph of $y = x^3$ and the graph of the function f. Describe the transformation from y to f.**

15. $f(x) = (x + 4)^3$ **16.** $f(x) = x^3 - 4$

17. $f(x) = -x^3 + 2$ **18.** $f(x) = (x + 3)^3 - 2$

19. $f(x) = -(x + 7)^3 - 2$ **20.** $f(x) = -(x - 1)^3 + 3$

Comparing End Behavior **In Exercises 21 and 22, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough to see the right-hand and left-hand behavior of each graph. Do the graphs of f and g have the same right-hand and left-hand behavior? Explain why or why not.**

21. $f(x) = \frac{1}{2}x^3 - 2x + 1$, $g(x) = \frac{1}{2}x^3$

22. $f(x) = -x^4 + 2x^3$, $g(x) = -x^4$

Applying the Leading Coefficient Test **In Exercises 23–26, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function.**

23. $f(x) = -x^2 + 6x + 9$

24. $f(x) = \frac{1}{2}x^3 + 2x$

25. $g(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

26. $h(x) = -x^5 - 7x^2 + 10x$

Finding Zeros of a Polynomial Function **In Exercises 27–32, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those in part (a).**

27. $g(x) = x^4 - x^3 - 2x^2$ **28.** $h(x) = -2x^3 - x^2 + x$

29. $f(t) = t^3 - 3t$ **30.** $f(x) = -(x + 6)^3 - 8$

31. $f(x) = x(x + 3)^2$ **32.** $f(t) = t^4 - 4t^2$

Finding a Polynomial Function with Given Zeros In Exercises 33–36, find a polynomial function that has the given zeros. (There are many correct answers.)

33. $-2, 1, 1, 5$

34. $-3, 0, 1, 4$

35. $3, 2 - \sqrt{3}, 2 + \sqrt{3}$

36. $-7, 4 - \sqrt{6}, 4 + \sqrt{6}$

Sketching the Graph of a Polynomial Function In Exercises 37 and 38, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

37. $f(x) = x^4 - 2x^3 - 12x^2 + 18x + 27$

38. $f(x) = 18 + 27x - 2x^2 - 3x^3$

Approximating the Zeros of a Function In Exercises 39–42, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your results in part (a) by using the *table* feature of the graphing utility.

39. $f(x) = x^3 + 2x^2 - x - 1$

40. $f(x) = 0.24x^3 - 2.6x - 1.4$

41. $f(x) = x^4 - 6x^2 - 4$

42. $f(x) = 2x^4 + \frac{7}{2}x^3 - 2$

2.3

Long Division of Polynomials In Exercises 43–50, use long division to divide.

43. $\dfrac{24x^2 - x - 8}{3x - 2}$

44. $\dfrac{4x^2 + 7}{3x - 2}$

45. $\dfrac{x^4 - 3x^2 + 2}{x^2 - 1}$

46. $\dfrac{3x^4 + x^2 - 1}{x^2 - 1}$

47. $(5x^3 - 13x^2 - x + 2) \div (x^2 - 3x + 1)$

48. $(x^4 + x^3 - x^2 + 2x) \div (x^2 + 2x)$

49. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

50. $\dfrac{x^4 - 3x^3 + 4x^2 - 6x + 3}{x^2 + 2}$

Using Synthetic Division In Exercises 51–56, use synthetic division to divide.

51. $(0.25x^4 - 4x^3) \div (x + 2)$

52. $(0.1x^3 + 0.3x^2 - 0.5) \div (x - 5)$

53. $(6x^4 - 4x^3 - 27x^2 + 18x) \div \left(x - \frac{2}{3}\right)$

54. $(2x^3 + 2x^2 - x + 2) \div \left(x - \frac{1}{2}\right)$

55. $(3x^3 - 10x^2 + 12x - 22) \div (x - 4)$

56. $(2x^3 + 6x^2 - 14x + 9) \div (x - 1)$

Using the Remainder Theorem In Exercises 57 and 58, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.

57. $f(x) = x^4 + 10x^3 - 24x^2 + 20x + 44$
(a) $f(-3)$ (b) $f(-2)$

58. $g(t) = 2t^5 - 5t^4 - 8t + 20$
(a) $g(-4)$ (b) $g(\sqrt{2})$

Factoring a Polynomial In Exercises 59–62, (a) verify the given factor(s) of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, and (d) list all real zeros of f. Confirm your results by using a graphing utility to graph the function.

	Function	*Factor(s)*
59.	$f(x) = x^3 + 4x^2 - 25x - 28$	$(x - 4)$
60.	$f(x) = 2x^3 + 11x^2 - 21x - 90$	$(x + 6)$
61.	$f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$	$(x + 2)$, $(x - 3)$
62.	$f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$	$(x - 2)$, $(x - 5)$

Using the Rational Zero Test In Exercises 63 and 64, use the Rational Zero Test to list all possible rational zeros of f. Use a graphing utility to verify that all the zeros of f are contained in the list.

63. $f(x) = 4x^3 - 11x^2 + 10x - 3$

64. $f(x) = 10x^3 + 21x^2 - x - 6$

Using Descartes's Rule of Signs In Exercises 65 and 66, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.

65. $g(x) = 5x^3 - 6x + 9$

66. $f(x) = 2x^5 - 3x^2 + 2x - 1$

Finding the Zeros of a Polynomial Function In Exercises 67 and 68, use synthetic division to verify the upper and lower bounds of the real zeros of f. Then find the real zeros of the function.

67. $f(x) = 4x^3 - 3x^2 + 4x - 3$
Upper bound: $x = 1$; Lower bound: $x = -\frac{1}{4}$

68. $f(x) = 2x^3 - 5x^2 - 14x + 8$
Upper bound: $x = 8$; Lower bound: $x = -4$

Finding the Zeros of a Polynomial Function In Exercises 69–72, find all the real zeros of the polynomial function.

69. $f(x) = 6x^3 + 31x^2 - 18x - 10$

70. $f(x) = x^3 - 1.3x^2 - 1.7x + 0.6$

71. $f(x) = 6x^4 - 25x^3 + 14x^2 + 27x - 18$

72. $f(x) = 5x^4 + 126x^2 + 25$

2.4

Writing a Complex Number in Standard Form **In Exercises 73–76, write the complex number in standard form.**

73. $6 + \sqrt{-25}$ **74.** $-\sqrt{-12} + 3$

75. $-2i^2 + 7i$ **76.** $-i^2 - 4i$

Operations with Complex Numbers **In Exercises 77–88, perform the operations and write the result in standard form.**

77. $(7 + 5i) + (-4 + 2i)$

78. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

79. $5i(13 - 8i)$ **80.** $(1 + 6i)(5 - 2i)$

81. $(10 - 8i)(2 - 3i)$ **82.** $i(6 + i)(3 - 2i)$

83. $(3 + 7i)^2 + (3 - 7i)^2$ **84.** $(4 - i)^2 - (4 + i)^2$

85. $(\sqrt{-16} + 3)(\sqrt{-25} - 2)$

86. $(5 - \sqrt{-4})(5 + \sqrt{-4})$

87. $\sqrt{-9} + 3 + \sqrt{-36}$

88. $7 - \sqrt{-81} + \sqrt{-49}$

Writing a Quotient of Complex Numbers in Standard Form **In Exercises 89–92, write the quotient in standard form.**

89. $\dfrac{6 + i}{i}$ **90.** $\dfrac{4}{-3i}$

91. $\dfrac{3 + 2i}{5 + i}$ **92.** $\dfrac{1 - 7i}{2 + 3i}$

Complex Solutions of a Quadratic Equation **In Exercises 93–98, solve the quadratic equation.**

93. $x^2 + 16 = 0$ **94.** $x^2 + 48 = 0$

95. $x^2 + 3x + 6 = 0$ **96.** $x^2 + 4x + 8 = 0$

97. $3x^2 - 5x + 6 = 0$ **98.** $5x^2 - 2x + 4 = 0$

2.5

Zeros of a Polynomial Function **In Exercises 99–102, confirm that the function has the indicated zero(s).**

99. $f(x) = x^2 + 6x + 9$; Repeated zero: -3

100. $f(x) = x^2 - 10x + 25$; Repeated zero: 5

101. $f(x) = x^3 + 16x$; $0, -4i, 4i$

102. $f(x) = x^3 + 144x$; $0, -12i, 12i$

Using the Factored Form of a Function **In Exercises 103 and 104, find all the zeros of the function.**

103. $f(x) = 3x(x - 2)^2$ **104.** $f(x) = (x - 4)(x + 9)^2$

Finding the Zeros of a Polynomial Function **In Exercises 105–110, find all the zeros of the function and write the polynomial as a product of linear factors. Verify your results by using a graphing utility to graph the function.**

105. $h(x) = x^3 - 7x^2 + 18x - 24$

106. $f(x) = 2x^3 - 5x^2 - 9x + 40$

107. $f(x) = 2x^4 - 5x^3 + 10x - 12$

108. $g(x) = 3x^4 - 4x^3 + 7x^2 + 10x - 4$

109. $f(x) = x^5 + x^4 + 5x^3 + 5x^2$

110. $f(x) = x^5 - 5x^3 + 4x$

Using the Zeros to Find the *x*-Intercepts **In Exercises 111–116, (a) find all the zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the *x*-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only *x*-intercepts.**

111. $f(x) = x^3 - 4x^2 + 6x - 4$

112. $f(x) = x^3 - 5x^2 - 7x + 51$

113. $f(x) = -3x^3 - 19x^2 - 4x + 12$

114. $f(x) = 2x^3 - 9x^2 + 22x - 30$

115. $f(x) = x^4 + 34x^2 + 225$

116. $f(x) = x^4 + 10x^3 + 26x^2 + 10x + 25$

Finding a Polynomial with Given Zeros **In Exercises 117–120, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

117. $4, -2, 5i$

118. $2, -2, 2i$

119. $1, -4, -3 + 5i$

120. $-4, -4, 1 + \sqrt{3}i$

Factoring a Polynomial **In Exercises 121 and 122, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.**

121. $f(x) = x^4 - 2x^3 + 8x^2 - 18x - 9$

(*Hint:* One factor is $x^2 + 9$.)

122. $f(x) = x^4 - 4x^3 + 3x^2 + 8x - 16$

(*Hint:* One factor is $x^2 - x - 4$.)

Finding the Zeros of a Polynomial Function **In Exercises 123 and 124, use the given zero to find all the zeros of the function.**

	Function	*Zero*
123.	$f(x) = x^3 + 3x^2 + 4x + 12$	$-2i$
124.	$f(x) = 2x^3 - 7x^2 + 14x + 9$	$2 + \sqrt{5}i$

2.6

Finding a Function's Domain and Asymptotes **In Exercises 125–136, (a) find the domain of the function, (b) decide whether the function is continuous, and (c) identify any horizontal and vertical asymptotes.**

125. $f(x) = \dfrac{2 - x}{x + 3}$ **126.** $f(x) = \dfrac{4x}{x - 8}$

127. $f(x) = \dfrac{2}{x^2 - 3x - 18}$ **128.** $f(x) = \dfrac{2x^2 + 3}{x^2 + x + 3}$

129. $f(x) = \dfrac{7 + x}{7 - x}$ **130.** $f(x) = \dfrac{6x}{x^2 - 1}$

131. $f(x) = \dfrac{4x^2}{2x^2 - 3}$ **132.** $f(x) = \dfrac{3x^2 - 11x - 4}{x^2 + 2}$

133. $f(x) = \dfrac{2x - 10}{x^2 - 2x - 15}$ **134.** $f(x) = \dfrac{x^3 - 4x^2}{x^2 + 3x + 2}$

135. $f(x) = \dfrac{x - 2}{|x| + 2}$ **136.** $f(x) = \dfrac{2x}{|2x - 1|}$

137. **Criminology** The cost C (in millions of dollars) for the U.S. government to seize $p\%$ of an illegal drug as it enters the country is given by

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the costs of seizing 25%, 50%, and 75% of the illegal drug.

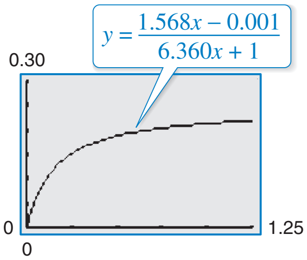
(b) Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window. Explain why you chose the values you used in your viewing window.

(c) According to this model, would it be possible to seize 100% of the drug? Explain.

138. **Biology** A biology class performs an experiment comparing the quantity of food consumed by a certain kind of moth with the quantity supplied. The model for the experimental data is given by

$$y = \frac{1.568x - 0.001}{6.360x + 1}, \quad x > 0$$

where x is the quantity (in milligrams) of food supplied and y is the quantity (in milligrams) eaten (see figure). At what level of consumption will the moth become satiated?



2.7

Finding Asymptotes and Holes **In Exercises 139–142, find all of the vertical, horizontal, and slant asymptotes, and any holes in the graph of the function. Then use a graphing utility to verify your result.**

139. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 1}$ **140.** $f(x) = \dfrac{2x^2 - 7x + 3}{2x^2 - 3x - 9}$

141. $f(x) = \dfrac{3x^2 + 5x - 2}{x + 1}$ **142.** $f(x) = \dfrac{2x^2 + 5x + 3}{x - 2}$

Sketching the Graph of a Rational Function **In Exercises 143–152, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes.**

143. $f(x) = \dfrac{2x - 1}{x - 5}$ **144.** $f(x) = \dfrac{x - 3}{x - 2}$

145. $f(x) = \dfrac{2x^2}{x^2 - 4}$ **146.** $f(x) = \dfrac{5x}{x^2 + 1}$

147. $f(x) = \dfrac{2}{(x + 1)^2}$ **148.** $f(x) = \dfrac{4}{(x - 1)^2}$

149. $f(x) = \dfrac{2x^3}{x^2 + 1}$ **150.** $f(x) = \dfrac{x^3}{3x^2 - 6}$

151. $f(x) = \dfrac{x^2 - x + 1}{x - 3}$ **152.** $f(x) = \dfrac{2x^2 + 7x + 3}{x + 1}$

153. **Biology** A Parks and Wildlife Commission releases 80,000 fish into a lake. After t years, the population N of the fish (in thousands) is given by

$$N = \frac{20(4 + 3t)}{1 + 0.05t}, \quad t \ge 0.$$

(a) Use a graphing utility to graph the function and find the populations when $t = 5$, $t = 10$, and $t = 25$.

(b) What is the maximum number of fish in the lake as time passes? Explain your reasoning.

154. **Publishing** A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are 2 inches deep and the margins on each side are 2 inches wide.

(a) Draw a diagram that illustrates the problem.

(b) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 7)}{x - 4}.$$

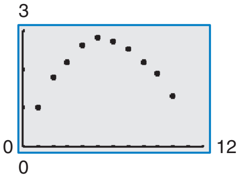
(c) Determine the domain of the function based on the physical constraints of the problem.

(d) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used.

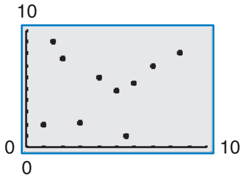
2.8

Classifying Scatter Plots In Exercises 155–158, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.

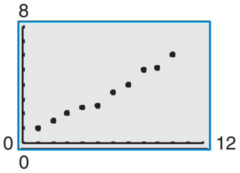
155.



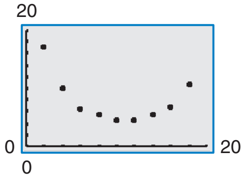
156.



157.



158.



159. MODELING DATA

The table shows the numbers of FM radio stations S in the United States from 2000 through 2009. (Source: Federal Communication Commission)

Year	FM stations, S
2000	5892
2001	6051
2002	6161
2003	6207
2004	6217
2005	6215
2006	6252
2007	6290
2008	6309
2009	6427

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) A cubic model for the data is

$$S = 2.520t^3 - 37.51t^2 + 192.4t + 5895.$$

Use the graphing utility to graph this model with the scatter plot from part (a).

(c) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

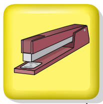
(d) Use the graphing utility to graph the quadratic model with the scatter plot from part (a).

(e) Which model is a better fit for the data? Explain.

(f) Use the model you chose in part (e) to predict the number of FM radio stations in 2012.

160. MODELING DATA

The table shows the sales S (in millions of dollars) of Office Depot for each of the years from 2002 through 2008. (Source: Office Depot)



Year	Sales, S (in millions of dollars)
2002	11,357
2003	12,359
2004	13,565
2005	14,279
2006	15,011
2007	15,528
2008	14,496

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

(d) According to the model, what is the first year when Office Depot will have sales of less than \$11 billion?

(e) Is this a good model for predicting the sales of Office Depot in future years? Explain.

Conclusions

True or False? In Exercises 161–163, determine whether the statement is true or false. Justify your answer.

161. The graph of $f(x) = \dfrac{2x^3}{x + 1}$ has a slant asymptote.

162. A fourth-degree polynomial with real coefficients can have -5, $-8i$, $4i$, and 5 as its zeros.

163. The sum of two complex numbers cannot be a real number.

164. Think About It Describe the domain restrictions of a rational function when the denominator divides evenly into the numerator.

165. Writing Write a paragraph discussing whether every rational function has a vertical asymptote.

166. Error Analysis Describe the error.

$$\xcancel{\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6}$$

167. Error Analysis Describe the error.

$$\xcancel{-i\left(\sqrt{-4} - 1\right) = -i(4i - 1) = -4i^2 + i = 4 + i}$$

168. Write each of the powers of i as i, $-i$, 1, or -1.

(a) i^{40} (b) i^{25} (c) i^{50} (d) i^{67}

2 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

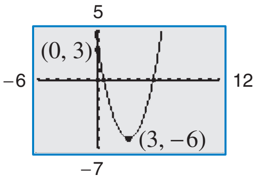


Figure for 2

1. Identify the vertex and intercepts of the graph of $y = x^2 + 4x + 3$.
2. Write an equation of the parabola shown at the right.
3. Find all the real zeros of $f(x) = 4x^3 + 4x^2 + x$. Determine the multiplicity of each zero.
4. Sketch the graph of the function $f(x) = -x^3 + 7x + 6$.
5. Divide using long division: $(3x^3 + 4x - 1) \div (x^2 + 1)$.
6. Divide using synthetic division: $(2x^4 - 5x^2 - 3) \div (x - 2)$.
7. Use synthetic division to evaluate $f(-2)$ for $f(x) = 3x^4 - 6x^2 + 5x - 1$.

In Exercises 8 and 9, list all the possible rational zeros of the function. Use a graphing utility to graph the function and find all the rational zeros.

8. $g(t) = 2t^4 - 3t^3 + 16t - 24$
9. $h(x) = 3x^5 + 2x^4 - 3x - 2$

10. Find all the zeros of the function $f(x) = x^3 - 7x^2 + 11x + 19$ and write the polynomial as a product of linear factors.

In Exercises 11–14, perform the operations and write the result in standard form.

11. $(-8 - 3i) + (-1 - 15i)$
12. $\left(10 + \sqrt{-20}\right) - \left(4 - \sqrt{-14}\right)$
13. $(2 + i)(6 - i)$
14. $(4 + 3i)^2 - (5 + i)^2$

In Exercises 15–17, write the quotient in standard form.

15. $\dfrac{8 + 5i}{6 - i}$
16. $\dfrac{5i}{2 + i}$
17. $(2i - 1) \div (3i + 2)$

In Exercises 18 and 19, solve the quadratic equation.

18. $x^2 + 75 = 0$
19. $x^2 - 2x + 8 = 0$

In Exercises 20–22, sketch the graph of the rational function. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and slant asymptotes.

20. $h(x) = \dfrac{4}{x^2} - 1$
21. $g(x) = \dfrac{x^2 + 2}{x - 1}$
22. $f(x) = \dfrac{2x^2 + 9}{5x^2 + 2}$

23. The table shows the amounts A (in billions of dollars) spent on military procurement by the Department of Defense for the years 2002 through 2008. (Source: U.S. Office of Management and Budget)
    (a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 2$ corresponding to 2002.
    (b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.
    (c) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?
    (d) Use the model to estimate the amounts spent on military procurement in 2010 and 2012.
    (e) Do you believe the model is useful for predicting the amounts spent on military procurement for years beyond 2008? Explain.

Year	Military procurement, A (in billions of dollars)
2002	62.5
2003	67.9
2004	76.2
2005	82.3
2006	89.8
2007	99.6
2008	117.4

Proofs in Mathematics

These two pages contain proofs of four important theorems about polynomial functions. The first two theorems are from Section 2.3, and the second two theorems are from Section 2.5.

The Remainder Theorem (p. 117)

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

Proof

From the Division Algorithm, you have

$$f(x) = (x - k)q(x) + r(x)$$

and because either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ must be a constant. That is, $r(x) = r$. Now, by evaluating $f(x)$ at $x = k$, you have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= (0)q(k) + r \\ &= r. \end{aligned}$$

To be successful in algebra, it is important that you understand the connection among the *factors* of a polynomial, the *zeros* of a polynomial function, and the *solutions* or *roots* of a polynomial equation. The Factor Theorem is the basis for this connection.

The Factor Theorem (p. 117)

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Proof

Using the Division Algorithm with the factor $(x - k)$, you have

$$f(x) = (x - k)q(x) + r(x).$$

By the Remainder Theorem, $r(x) = r = f(k)$, and you have

$$f(x) = (x - k)q(x) + f(k)$$

where $q(x)$ is a polynomial of lesser degree than $f(x)$. If $f(k) = 0$, then

$$f(x) = (x - k)q(x)$$

and you see that $(x - k)$ is a factor of $f(x)$. Conversely, if $(x - k)$ is a factor of $f(x)$, then division of $f(x)$ by $(x - k)$ yields a remainder of 0. So, by the Remainder Theorem, you have $f(k) = 0$.

Linear Factorization Theorem (p. 135)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Proof

Using the Fundamental Theorem of Algebra, you know that f must have at least one zero, c_1. Consequently, $(x - c_1)$ is a factor of $f(x)$, and you have

$$f(x) = (x - c_1)f_1(x).$$

If the degree of $f_1(x)$ is greater than zero, then apply the Fundamental Theorem again to conclude that f_1 must have a zero c_2, which implies that

$$f(x) = (x - c_1)(x - c_2)f_2(x).$$

It is clear that the degree of $f_1(x)$ is $n - 1$, that the degree of $f_2(x)$ is $n - 2$, and that you can repeatedly apply the Fundamental Theorem n times until you obtain

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where a_n is the leading coefficient of the polynomial $f(x)$.

Factors of a Polynomial (p. 138)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Proof

To begin, you use the Linear Factorization Theorem to conclude that $f(x)$ can be *completely* factored in the form

$$f(x) = d(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n).$$

If each c_i is real, then there is nothing more to prove. If any c_i is complex ($c_i = a + bi$, $b \neq 0$), then, because the coefficients of $f(x)$ are real, you know that the conjugate $c_j = a - bi$ is also a zero. By multiplying the corresponding factors, you obtain

$$\begin{aligned}(x - c_i)(x - c_j) &= [x - (a + bi)][x - (a - bi)] \\ &= x^2 - 2ax + (a^2 + b^2)\end{aligned}$$

where each coefficient is real.

The Fundamental Theorem of Algebra

The Linear Factorization Theorem is closely related to the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra has a long and interesting history. In the early work with polynomial equations, the Fundamental Theorem of Algebra was thought to have been not true, because imaginary solutions were not considered. In fact, in the very early work by mathematicians such as Abu al-Khwarizmi (c. 800 A.D.), negative solutions were also not considered.

Once imaginary numbers were accepted, several mathematicians attempted to give a general proof of the Fundamental Theorem of Algebra. These included Gottfried von Leibniz (1702), Jean d'Alembert (1746), Leonhard Euler (1749), Joseph-Louis Lagrange (1772), and Pierre Simon Laplace (1795). The mathematician usually credited with the first correct proof of the Fundamental Theorem of Algebra is Carl Friedrich Gauss, who published the proof in his doctoral thesis in 1799.

Progressive Summary (Chapters 1–2)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, and 9. In each Progressive Summary, new topics encountered for the first time appear in red.

ALGEBRAIC FUNCTIONS

Polynomial, Rational, Radical

■ Rewriting

Polynomial form ↔ Factored form
Operations with polynomials
Rationalize denominators
Simplify rational expressions
Operations with complex numbers

■ Solving

Equation	*Strategy*
Linear	Isolate variable
Quadratic	Factor, set to zero Extract square roots Complete the square Quadratic Formula
Polynomial	Factor, set to zero Rational Zero Test
Rational	Multiply by LCD
Radical	Isolate, raise to power
Absolute value	Isolate, form two equations

■ Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Symmetry	Transformations
Slope	Composition
Asymptotes	Standard forms of equations
End behavior	Leading Coefficient Test
Minimum values	Synthetic division
Maximum values	Descartes's Rule of Signs

Numerically

Table of values

TRANSCENDENTAL FUNCTIONS

■ Rewriting

■ Solving

■ Analyzing

OTHER TOPICS

■ Rewriting

■ Solving

■ Analyzing