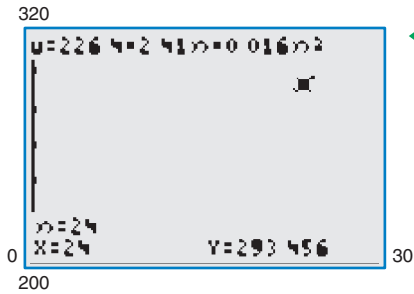


8

Sequences, Series, and Probability



- 8.1 Sequences and Series
- 8.2 Arithmetic Sequences and Partial Sums
- 8.3 Geometric Sequences and Series
- 8.4 The Binomial Theorem
- 8.5 Counting Principles
- 8.6 Probability

Section 8.1, Example 10
Population of the United States



8.1 Sequences and Series

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English. Saying that a collection is listed *in sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers. Instead of using function notation, sequences are usually written using subscript notation, as shown in the following definition.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

are the **terms** of the sequence. If the domain of a function consists of the first n positive integers only, then the sequence is a **finite sequence**.

On occasion, it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become

$$a_0, a_1, a_2, a_3, \dots$$

The domain of the function is the set of nonnegative integers.

Example 1 Writing the Terms of a Sequence

Write the first four terms of each sequence.

a. $a_n = 3n - 2$ b. $a_n = 3 + (-1)^n$

Solution

a. The first four terms of the sequence given by

$$a_n = 3n - 2$$

are

$$a_1 = 3(1) - 2 = 1 \quad \text{1st term}$$

$$a_2 = 3(2) - 2 = 4 \quad \text{2nd term}$$

$$a_3 = 3(3) - 2 = 7 \quad \text{3rd term}$$

$$a_4 = 3(4) - 2 = 10. \quad \text{4th term}$$

b. The first four terms of the sequence given by

$$a_n = 3 + (-1)^n$$

are

$$a_1 = 3 + (-1)^1 = 3 - 1 = 2 \quad \text{1st term}$$

$$a_2 = 3 + (-1)^2 = 3 + 1 = 4 \quad \text{2nd term}$$

$$a_3 = 3 + (-1)^3 = 3 - 1 = 2 \quad \text{3rd term}$$

$$a_4 = 3 + (-1)^4 = 3 + 1 = 4. \quad \text{4th term}$$

What you should learn

- Use sequence notation to write the terms of sequences.
- Use factorial notation.
- Use summation notation to write sums.
- Find sums of infinite series.
- Use sequences and series to model and solve real-life problems.

Why you should learn it

Sequences and series are useful in modeling sets of values in order to identify patterns. For instance, Exercise 122 on page 579 shows how a sequence can be used to model the population of trout in a lake.



CHECKPOINT Now try Exercise 7.

Example 2 Writing the Terms of a Sequence

Write the first five terms of the sequence given by

$$a_n = \frac{(-1)^n}{2n - 1}.$$

Algebraic Solution

The first five terms of the sequence are as follows.

$$a_1 = \frac{(-1)^1}{2(1) - 1} = \frac{-1}{2 - 1} = -1 \quad \text{1st term}$$

$$a_2 = \frac{(-1)^2}{2(2) - 1} = \frac{1}{4 - 1} = \frac{1}{3} \quad \text{2nd term}$$

$$a_3 = \frac{(-1)^3}{2(3) - 1} = \frac{-1}{6 - 1} = -\frac{1}{5} \quad \text{3rd term}$$

$$a_4 = \frac{(-1)^4}{2(4) - 1} = \frac{1}{8 - 1} = \frac{1}{7} \quad \text{4th term}$$

$$a_5 = \frac{(-1)^5}{2(5) - 1} = \frac{-1}{10 - 1} = -\frac{1}{9} \quad \text{5th term}$$

 **CHECKPOINT** Now try Exercise 17.

Simply listing the first few terms is not sufficient to define a unique sequence—the n th term *must be given*. To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{2^n}, \dots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \dots, \frac{6}{(n+1)(n^2 - n + 6)}, \dots$$

Example 3 Finding the n th Term of a Sequence

Write an expression for the apparent n th term a_n of each sequence.

a. 1, 3, 5, 7, . . .

b. 2, 5, 10, 17, . . .

Solution

a. n : 1 2 3 4 . . . n

Terms: 1 3 5 7 . . . a_n

Apparent Pattern: Each term is 1 less than twice n . So, the apparent n th term is

$$a_n = 2n - 1.$$

b. n : 1 2 3 4 . . . n

Terms: 2 5 10 17 . . . a_n

Apparent Pattern: Each term is 1 more than the square of n . So, the apparent n th term is

$$a_n = n^2 + 1.$$

 **CHECKPOINT** Now try Exercise 39.

Numerical Solution

Set your graphing utility to *sequence* mode and enter the sequence. Use the *table* feature to create a table showing the terms of the sequence, as shown in Figure 8.1.

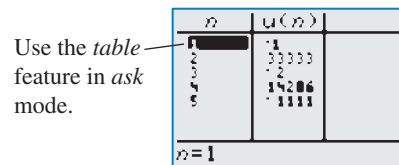


Figure 8.1

So, you can estimate the first five terms of the sequence as follows.

$$u_1 = -1, \quad u_2 = 0.33333 \approx \frac{1}{3}, \quad u_3 = -0.2 = -\frac{1}{5},$$

$$u_4 = 0.14286 \approx \frac{1}{7}, \quad \text{and} \quad u_5 = -0.11111 \approx -\frac{1}{9}$$

Some sequences are defined **recursively**. To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms.

Example 4 A Recursive Sequence

A sequence is defined recursively as follows.

$$a_1 = 3, a_k = 2a_{k-1} + 1, \text{ where } k \geq 2$$

Write the first five terms of this sequence.

Solution

- $a_1 = 3$ 1st term is given.
- $a_2 = 2a_{2-1} + 1 = 2a_1 + 1 = 2(3) + 1 = 7$ Use recursion formula.
- $a_3 = 2a_{3-1} + 1 = 2a_2 + 1 = 2(7) + 1 = 15$ Use recursion formula.
- $a_4 = 2a_{4-1} + 1 = 2a_3 + 1 = 2(15) + 1 = 31$ Use recursion formula.
- $a_5 = 2a_{5-1} + 1 = 2a_4 + 1 = 2(31) + 1 = 63$ Use recursion formula.

CHECKPOINT Now try Exercise 53.

In the next example you will study a well-known recursive sequence, the Fibonacci sequence.

Example 5 The Fibonacci Sequence: A Recursive Sequence

The Fibonacci sequence is defined recursively as follows.

$$a_0 = 1, a_1 = 1, a_k = a_{k-2} + a_{k-1}, \text{ where } k \geq 2$$

Write the first six terms of this sequence.

Solution

- $a_0 = 1$ 0th term is given.
- $a_1 = 1$ 1st term is given.
- $a_2 = a_{2-2} + a_{2-1} = a_0 + a_1 = 1 + 1 = 2$ Use recursion formula.
- $a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 2 = 3$ Use recursion formula.
- $a_4 = a_{4-2} + a_{4-1} = a_2 + a_3 = 2 + 3 = 5$ Use recursion formula.
- $a_5 = a_{5-2} + a_{5-1} = a_3 + a_4 = 3 + 5 = 8$ Use recursion formula.

You can check this result using the *table* feature of a graphing utility, as shown in Figure 8.2.

n	$u(n)$
0	1
1	1
2	2
3	3
4	5
5	8

Figure 8.2

CHECKPOINT Now try Exercise 57.

Technology Tip



To graph a sequence using a graphing utility, set the mode to *dot* and *sequence* and enter the sequence. Try graphing the sequence in Example 4 and using the *trace* feature to identify the terms. For instructions on how to use the sequence mode, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

```
Plot1 Plot2 Plot3
nMin=1
u(n)=2u(n-1)+1
u(nMin)=3
v(n)=
v(nMin)=
w(n)=
w(nMin)=
```



Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials**.

Definition of Factorial

If n is a positive integer, then n **factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

Notice that $0! = 1$ and $1! = 1$. Here are some other values of $n!$.

$$2! = 1 \cdot 2 = 2 \quad 3! = 1 \cdot 2 \cdot 3 = 6 \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

The value of n does not have to be very large before the value of $n!$ becomes huge. For instance, $10! = 3,628,800$.

Factorials follow the same conventions for order of operations as do exponents. For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n)$$

whereas $(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot 2n$.

Example 6 Writing the Terms of a Sequence Involving Factorials

Write the first five terms of the sequence given by $a_n = \frac{2^n}{n!}$. Begin with $n = 0$.

Algebraic Solution

$$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1 \quad \text{0th term}$$

$$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2 \quad \text{1st term}$$

$$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2 \quad \text{2nd term}$$

$$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3} \quad \text{3rd term}$$

$$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3} \quad \text{4th term}$$

 **CHECKPOINT** Now try Exercise 63.

Example 7 Simplifying Factorial Expressions

$$\text{a. } \frac{8!}{2! \cdot 6!} = \frac{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6} \cdot 7 \cdot 8}{1 \cdot 2 \cdot 1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6}} = \frac{7 \cdot 8}{2} = 28$$

$$\text{b. } \frac{n!}{(n-1)!} = \frac{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cdots \cdot \cancel{(n-1)} \cdot n}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cdots \cdot \cancel{(n-1)}} = n$$

 **CHECKPOINT** Now try Exercise 73.

Explore the Concept



Most graphing utilities have the capability to compute $n!$. Use your graphing utility to compare $3 \cdot 5!$ and $(3 \cdot 5)!$. How do they differ? How large a value of $n!$ will your graphing utility allow you to compute?

Graphical Solution

Using a graphing utility set to *dot* and *sequence* modes, enter the sequence. Next, graph the sequence, as shown in Figure 8.3.

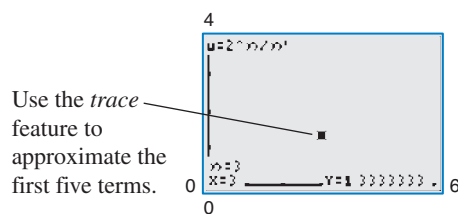


Figure 8.3

You can estimate the first five terms of the sequence as follows.

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 2, \\ u_3 \approx 1.333 \approx \frac{4}{3}, \quad u_4 \approx 0.667 \approx \frac{2}{3}$$

Study Tip



In Example 7(a), you can also simplify the computation as

$$\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot \cancel{6!}}{2! \cdot \cancel{6!}} = 28.$$

Summation Notation

There is a convenient notation for the sum of the terms of a finite sequence. It is called **summation notation** or **sigma notation** because it involves the use of the uppercase Greek letter sigma, written as Σ .

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

where i is called the **index of summation**, n is the **upper limit of summation**, and 1 is the **lower limit of summation**.

Example 8 Sigma Notation for Sums

$$\begin{aligned} \text{a. } \sum_{i=1}^5 3i &= 3(1) + 3(2) + 3(3) + 3(4) + 3(5) \\ &= 3(1 + 2 + 3 + 4 + 5) \\ &= 3(15) \\ &= 45 \end{aligned}$$

$$\begin{aligned} \text{b. } \sum_{k=3}^6 (1 + k^2) &= (1 + 3^2) + (1 + 4^2) + (1 + 5^2) + (1 + 6^2) \\ &= 10 + 17 + 26 + 37 \\ &= 90 \end{aligned}$$

$$\begin{aligned} \text{c. } \sum_{n=0}^8 \frac{1}{n!} &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40,320} \\ &\approx 2.71828 \end{aligned}$$

For the summation in part (c), note that the sum is very close to the irrational number $e \approx 2.718281828$. It can be shown that as more terms of the sequence whose n th term is $1/n!$ are added, the sum becomes closer and closer to e .

 **CHECKPOINT** Now try Exercise 87.

In Example 8, note that the lower limit of a summation does not have to be 1. Also note that the index of summation does not have to be the letter i . For instance, in part (b), the letter k is the index of summation.

Properties of Sums (See the proofs on page 632.)

$$1. \sum_{i=1}^n c = cn, \quad c \text{ is a constant.}$$

$$2. \sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i, \quad c \text{ is a constant.}$$

$$3. \sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$4. \sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

Study Tip



Summation notation is an instruction to add the terms of a sequence. From the definition at the left, the upper limit of summation tells you where to end the sum. Summation notation helps you generate the appropriate terms of the sequence prior to finding the actual sum, which may be unclear.

Study Tip



Variations in the upper and lower limits of summation can produce quite different-looking summation notations for *the same sum*. For example, the following two sums have identical terms.

$$\sum_{i=1}^3 3(2^i) = 3(2^1 + 2^2 + 2^3)$$

$$\sum_{i=0}^2 3(2^{i+1}) = 3(2^1 + 2^2 + 2^3)$$

Series

Many applications involve the sum of the terms of a finite or an infinite sequence. Such a sum is called a **series**.

Definition of a Series

Consider the infinite sequence $a_1, a_2, a_3, \dots, a_i, \dots$

1. The sum of the first n terms of the sequence is called a **finite series** or the **partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

Example 9 Finding the Sum of a Series

For the series

$$\sum_{i=1}^{\infty} \frac{3}{10^i}$$

find (a) the third partial sum and (b) the sum.

Solution

- a. The third partial sum is

$$\begin{aligned} \sum_{i=1}^3 \frac{3}{10^i} &= \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} \\ &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} \\ &= 0.3 + 0.03 + 0.003 \\ &= 0.333. \end{aligned}$$

- b. The sum of the series is

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{3}{10^i} &= \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \cdots \\ &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \frac{3}{100,000} + \cdots \\ &= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \\ &= 0.33333 \dots \\ &= \frac{1}{3}. \end{aligned}$$

 **CHECKPOINT** Now try Exercise 117.

Notice in Example 9(b) that the sum of an infinite series can be a finite number.

Technology Tip



Most graphing utilities are able to sum the first n terms of a sequence. Try using a graphing utility to confirm the results in Example 8 and Example 9(a).

```
sum(seq(1/n!,n,0,8))
2.71827877
```

Application

Sequences have many applications in situations that involve recognizable patterns. One such model is illustrated in Example 10.

Example 10 Population of the United States



From 1980 through 2008, the resident population of the United States can be approximated by the model

$$a_n = 226.4 + 2.41n + 0.016n^2, \quad n = 0, 1, \dots, 28$$

where a_n is the population (in millions) and n represents the year, with $n = 0$ corresponding to 1980. Find the last five terms of this finite sequence. (Source: U.S. Census Bureau)

Algebraic Solution

The last five terms of this finite sequence are as follows.

$$\begin{aligned} a_{24} &= 226.4 + 2.41(24) + 0.016(24)^2 \\ &\approx 293.5 && \text{2004 population} \end{aligned}$$

$$\begin{aligned} a_{25} &= 226.4 + 2.41(25) + 0.016(25)^2 \\ &\approx 296.7 && \text{2005 population} \end{aligned}$$

$$\begin{aligned} a_{26} &= 226.4 + 2.41(26) + 0.016(26)^2 \\ &\approx 299.9 && \text{2006 population} \end{aligned}$$

$$\begin{aligned} a_{27} &= 226.4 + 2.41(27) + 0.016(27)^2 \\ &\approx 303.1 && \text{2007 population} \end{aligned}$$

$$\begin{aligned} a_{28} &= 226.4 + 2.41(28) + 0.016(28)^2 \\ &\approx 306.4 && \text{2008 population} \end{aligned}$$

CHECKPOINT Now try Exercise 121.

Graphical Solution

Using a graphing utility set to *dot* and *sequence* modes, enter the sequence. Next, graph the sequence, as shown in Figure 8.4.

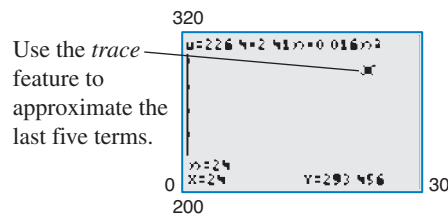


Figure 8.4

You can estimate the last five terms of the sequence as follows.

$$\begin{aligned} a_{24} &\approx 293.5, & a_{25} &\approx 296.7, & a_{26} &\approx 299.9, \\ a_{27} &\approx 303.1, & a_{28} &\approx 306.4 \end{aligned}$$

Explore the Concept



A $3 \times 3 \times 3$ cube is created using 27 unit cubes (a unit cube has a length, width, and height of 1 unit), and only the faces of the cubes that are visible are painted blue (see Figure 8.5). Complete the table below to determine how many unit cubes of the $3 \times 3 \times 3$ cube have no blue faces, one blue face, two blue faces, and three blue faces. Do the same for a $4 \times 4 \times 4$ cube, a $5 \times 5 \times 5$ cube, and a $6 \times 6 \times 6$ cube, and record your results in the table below. What type of pattern do you observe in the table? Write a formula you could use to determine the column values for an $n \times n \times n$ cube.

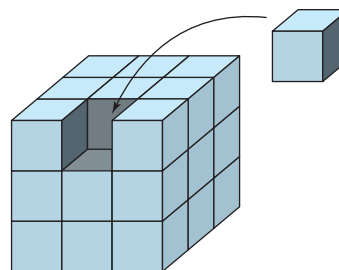


Figure 8.5

Cube	Number of blue faces			
	0	1	2	3
$3 \times 3 \times 3$				

8.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

- The function values $a_1, a_2, a_3, a_4, \dots, a_n, \dots$ are called the _____ of a sequence.
- If you are given one or more of the first few terms of a sequence, and all other terms of the sequence are defined using previous terms, then the sequence is defined _____.
- For the sum $\sum_{i=1}^n a_i$, i is called the _____ of summation, n is the _____ of summation, and 1 is the _____ of summation.
- The sum of the terms of a finite or an infinite sequence is called a _____.
- Which describes an infinite sequence? a finite sequence?
 - The domain consists of the first n positive integers.
 - The domain consists of the set of positive integers.
- Write $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$ in factorial notation.

Procedures and Problem Solving

Writing the Terms of a Sequence In Exercises 7–16, write the first five terms of the sequence. (Assume n begins with 1.)

- | | |
|---|--|
| ✓ 7. $a_n = 2n + 5$ | 8. $a_n = 4n - 7$ |
| 9. $a_n = 3^n$ | 10. $a_n = \left(\frac{1}{2}\right)^n$ |
| 11. $a_n = \left(-\frac{1}{2}\right)^n$ | 12. $a_n = (-2)^n$ |
| 13. $a_n = \frac{n+1}{n}$ | 14. $a_n = \frac{n}{n+1}$ |
| 15. $a_n = \frac{n}{n^2+1}$ | 16. $a_n = \frac{2n}{n+1}$ |

Writing the Terms of a Sequence In Exercises 17–26, write the first five terms of the sequence (a) using the *table* feature of a graphing utility and (b) algebraically. (Assume n begins with 1.)

- | | |
|------------------------------------|---|
| ✓ 17. $a_n = \frac{1 + (-1)^n}{n}$ | 18. $a_n = \frac{1 + (-1)^n}{2n}$ |
| 19. $a_n = 1 - \frac{1}{2^n}$ | 20. $a_n = \frac{3^n}{4^n}$ |
| 21. $a_n = \frac{1}{n^{3/2}}$ | 22. $a_n = \frac{1}{\sqrt{n}}$ |
| 23. $a_n = \frac{(-1)^n}{n^2}$ | 24. $a_n = (-1)^n \left(\frac{n}{n+1}\right)$ |
| 25. $a_n = (2n-1)(2n+1)$ | 26. $a_n = n(n-1)(n-2)$ |

Using a Graphing Utility In Exercises 27–32, use the *table* feature of a graphing utility to find the first 10 terms of the sequence. (Assume n begins with 1.)

27. $a_n = 2(3n-1) + 5$ 28. $a_n = 2n(n+1)(n+2)$

29. $a_n = 1 + \frac{n+1}{n}$

30. $a_n = \frac{4n^2}{n+2}$

31. $a_n = (-1)^n + 1$

32. $a_n = (-1)^{n+1} + 1$

Writing an Indicated Term In Exercises 33–38, find the indicated term of the sequence.

33. $a_n = \frac{n^2}{n^2+1}$

34. $a_n = \frac{n^2}{2n+1}$

$a_{10} = \square$

$a_5 = \square$

35. $a_n = (-1)^n(3n-2)$

36. $a_n = (-1)^{n-1}[n(n-1)]$

$a_{25} = \square$

$a_{16} = \square$

37. $a_n = \frac{2^n}{2^n+1}$

38. $a_n = \frac{2^{n+1}}{2^n+1}$

$a_6 = \square$

$a_7 = \square$

Finding the n th Term of a Sequence In Exercises 39–52, write an expression for the *apparent* n th term of the sequence. (Assume n begins with 1.)

✓ 39. 1, 4, 7, 10, 13, . . . 40. 3, 7, 11, 15, 19, . . .

41. 0, 3, 8, 15, 24, . . . 42. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$

43. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \dots$ 44. $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \dots$

45. $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \dots$ 46. $\frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \dots$

47. $1 + \frac{1}{1}, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \dots$

48. $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \dots$

49. $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots$ 50. $1, 2, \frac{2^2}{2}, \frac{2^3}{6}, \frac{2^4}{24}, \frac{2^5}{120}, \dots$

51. 1, 3, 1, 3, 1, . . .

52. 1, -1, 1, -1, 1, . . .

A Recursive Sequence In Exercises 53–58, write the first five terms of the sequence defined recursively.

- ✓ 53. $a_1 = 28, a_{k+1} = a_k - 4$
- 54. $a_1 = 15, a_{k+1} = a_k + 3$
- 55. $a_1 = 3, a_{k+1} = 2(a_k - 1)$
- 56. $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$
- ✓ 57. $a_0 = 1, a_1 = 3, a_k = a_{k-2} + a_{k-1}$
- 58. $a_0 = -1, a_1 = 1, a_k = a_{k-2} + a_{k-1}$

Finding the n th Term of a Recursive Sequence In Exercises 59–62, write the first five terms of the sequence defined recursively. Use the pattern to write the n th term of the sequence as a function of n . (Assume n begins with 1.)

- 59. $a_1 = 6, a_{k+1} = a_k + 2$
- 60. $a_1 = 25, a_{k+1} = a_k - 5$
- 61. $a_1 = 81, a_{k+1} = \frac{1}{3}a_k$
- 62. $a_1 = 14, a_{k+1} = -2a_k$

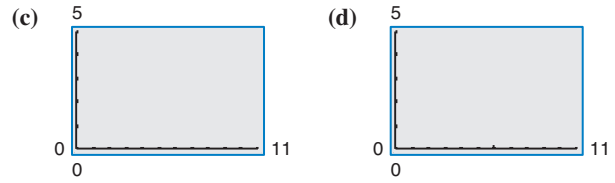
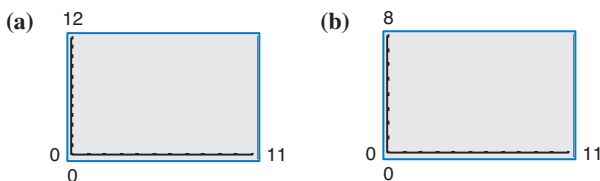
Writing the Terms of a Sequence Involving Factorials In Exercises 63–68, write the first five terms of the sequence (a) using the *table* feature of a graphing utility and (b) algebraically. (Assume n begins with 0.)

- ✓ 63. $a_n = \frac{1}{n!}$
- 64. $a_n = \frac{1}{(n+1)!}$
- 65. $a_n = \frac{n!}{2n+1}$
- 66. $a_n = \frac{n^2}{(n+1)!}$
- 67. $a_n = \frac{(-1)^{2n}}{(2n)!}$
- 68. $a_n = \frac{(-1)^{2n+1}}{(2n+1)!}$

Simplifying Factorial Expressions In Exercises 69–76, simplify the factorial expression.

- 69. $\frac{2!}{4!}$
- 70. $\frac{5!}{7!}$
- 71. $\frac{12!}{4! \cdot 8!}$
- 72. $\frac{10! \cdot 3!}{4! \cdot 6!}$
- ✓ 73. $\frac{(n+1)!}{n!}$
- 74. $\frac{(n+2)!}{n!}$
- 75. $\frac{(2n-1)!}{(2n+1)!}$
- 76. $\frac{(2n+2)!}{(2n)!}$

Identifying a Graph In Exercises 77–80, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]



- 77. $a_n = \frac{8}{n+1}$
- 78. $a_n = \frac{8n}{n+1}$
- 79. $a_n = 4(0.5)^{n-1}$
- 80. $a_n = \frac{4^n}{n!}$

Graphing the Terms of a Sequence In Exercises 81–86, use a graphing utility to graph the first 10 terms of the sequence. (Assume n begins with 1.)

- 81. $a_n = \frac{2}{3}n$
- 82. $a_n = 2 - \frac{4}{n}$
- 83. $a_n = 16(-0.5)^{n-1}$
- 84. $a_n = 8(0.75)^{n-1}$
- 85. $a_n = \frac{2n}{n+1}$
- 86. $a_n = \frac{3n^2}{n^2+1}$

Sigma Notation for Sums In Exercises 87–98, find the sum.

- ✓ 87. $\sum_{i=1}^5 (2i+1)$
- 88. $\sum_{i=1}^6 (3i-1)$
- 89. $\sum_{k=1}^4 10$
- 90. $\sum_{k=1}^5 4$
- 91. $\sum_{i=0}^4 i^2$
- 92. $\sum_{i=0}^5 3i^2$
- 93. $\sum_{k=0}^3 \frac{1}{k^2+1}$
- 94. $\sum_{j=3}^5 \frac{1}{j}$
- 95. $\sum_{i=1}^4 [(i-1)^2 + (i+1)^3]$
- 96. $\sum_{k=2}^5 (k+1)(k-3)$
- 97. $\sum_{i=1}^4 2^i$
- 98. $\sum_{j=0}^4 (-2)^j$

Using a Graphing Utility In Exercises 99–102, use a graphing utility to find the sum.

- 99. $\sum_{j=1}^6 (24-3j)$
- 100. $\sum_{j=1}^{10} \frac{3}{j+1}$
- 101. $\sum_{k=0}^4 \frac{(-1)^k}{k+1}$
- 102. $\sum_{k=0}^4 \frac{(-1)^k}{k!}$

Writing a Sum Using Sigma Notation In Exercises 103–112, use sigma notation to write the sum. Then use a graphing utility to find the sum.

- 103. $\frac{1}{3(1)} + \frac{1}{3(2)} + \frac{1}{3(3)} + \cdots + \frac{1}{3(9)}$
- 104. $\frac{5}{1+1} + \frac{5}{1+2} + \frac{5}{1+3} + \cdots + \frac{5}{1+15}$

105. $\left[2\left(\frac{1}{8}\right) + 3\right] + \left[2\left(\frac{2}{8}\right) + 3\right] + \cdots + \left[2\left(\frac{8}{8}\right) + 3\right]$
 106. $\left[1 - \left(\frac{1}{6}\right)^2\right] + \left[1 - \left(\frac{2}{6}\right)^2\right] + \cdots + \left[1 - \left(\frac{6}{6}\right)^2\right]$
 107. $3 - 9 + 27 - 81 + 243 - 729$
 108. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128}$
 109. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots - \frac{1}{20^2}$
 110. $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{10 \cdot 12}$
 111. $\frac{1}{4} + \frac{3}{8} + \frac{7}{16} + \frac{15}{32} + \frac{31}{64}$
 112. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \frac{720}{64}$

Finding a Partial Sum In Exercises 113–116, find the indicated partial sum of the series.

113. $\sum_{i=1}^{\infty} 5\left(\frac{1}{2}\right)^i$ Fourth partial sum
 114. $\sum_{i=1}^{\infty} 2\left(\frac{1}{3}\right)^i$ Fifth partial sum
 115. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^n$ Third partial sum
 116. $\sum_{n=1}^{\infty} 8\left(-\frac{1}{4}\right)^n$ Fourth partial sum

Finding the Sum of a Series In Exercises 117–120, find (a) the fourth partial sum and (b) the sum of the infinite series.

- ✓ 117. $\sum_{i=1}^{\infty} 6\left(\frac{1}{10}\right)^i$ 118. $\sum_{k=1}^{\infty} 4\left(\frac{1}{10}\right)^k$
 119. $\sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^k$ 120. $\sum_{i=1}^{\infty} 2\left(\frac{1}{10}\right)^i$

- ✓ 121. **Compound Interest** A deposit of \$5000 is made in an account that earns 3% interest compounded quarterly. The balance in the account after n quarters is given by

$$A_n = 5000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 1, 2, 3, \dots$$

- (a) Compute the first eight terms of this sequence.
 (b) Find the balance in this account after 10 years by computing the 40th term of the sequence.

122. **Why you should learn it** (p. 570) A landlocked lake has been selected to be stocked in the year 2012 with 5500 trout, and to be restocked each year thereafter with 500 trout. Each year the fish population declines 25% due to harvesting and other natural causes.



- (a) Write a recursive sequence that gives the population p_n of trout in the lake in terms of the year n since stocking began.
 (b) Use the recursion formula from part (a) to find the numbers of trout in the lake in the years 2013, 2014, and 2015.

Ljupco Smokovski 2010/used under license from Shutterstock.com
 Rob Wilson 2010/used under license from Shutterstock.com

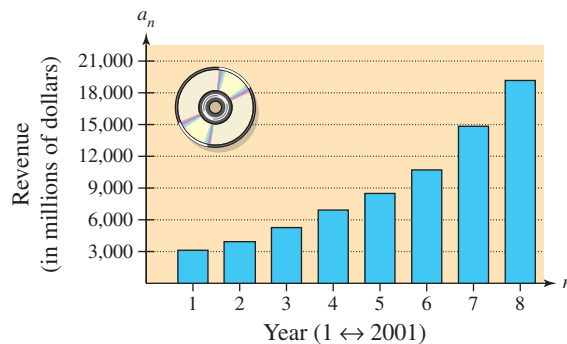
- (c) Use a graphing utility to find the number of trout in the lake as time passes infinitely. Explain your result.

123. **Business** The revenues a_n (in millions of dollars) of Amazon.com from 2001 through 2008 are shown in the bar graph. The revenues can be approximated by the model

$$a_n = 296.477n^2 - 469.11n + 3606.2$$

$$n = 1, 2, \dots, 8$$

where n is the year, with $n = 1$ corresponding to 2001. Use this model to approximate the total revenue from 2001 through 2008. Compare this sum with the result of adding the revenues shown in the bar graph. (Source: Amazon.com)



124. MODELING DATA

The numbers R_n (in thousands) of registrations of publicly owned buses in the United States from 2003 through 2008 are shown in the table. (Source: U.S. Federal Highway Administration)

Year	Registrations, R_n (in thousands)
2003	452
2004	465
2005	476
2006	483
2007	490
2008	493



- (a) Use a graphing utility to graph the data. Let n represent the year, with $n = 3$ corresponding to 2003.
 (b) Use the *regression* feature of the graphing utility to find a linear sequence and a quadratic sequence that model the data. Identify the coefficient of determination for each model.
 (c) Graph each model with the data. Decide which model is a better fit for the data. Explain.
 (d) Use the model you chose in part (c) to predict the number of bus registrations in 2015. Does your answer seem reasonable? Explain.

Conclusions

True or False? In Exercises 125 and 126, determine whether the statement is true or false. Justify your answer.

125. $\sum_{i=1}^4 (i^2 + 2i) = \sum_{i=1}^4 i^2 + 2 \sum_{i=1}^4 i$

126. $\sum_{j=1}^4 2^j = \sum_{j=3}^6 2^{j-2}$

Fibonacci Sequence In Exercises 127 and 128, use the Fibonacci sequence. (See Example 5.)

127. Write the first 12 terms of the Fibonacci sequence a_n and the first 10 terms of the sequence given by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n > 0.$$

128. Using the definition of b_n given in Exercise 127, show that b_n can be defined recursively by

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

Exploration In Exercises 129–132, let

$$a_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

be a sequence with n th term a_n .

129. Use the *table* feature of a graphing utility to find the first five terms of the sequence.

130. Do you recognize the terms of the sequence in Exercise 129? What sequence is it?

131. Find expressions for a_{n+1} and a_{n+2} in terms of n .

132. Use the result from Exercise 131 to show that $a_{n+2} = a_{n+1} + a_n$. Is this result the same as your answer to Exercise 129? Explain.

f A Sequence Involving x In Exercises 133–142, write the first five terms of the sequence.

133. $a_n = \frac{x^n}{n!}$

134. $a_n = \frac{x^2}{n^2}$

135. $a_n = \frac{(-1)^n x^{2n+1}}{2n+1}$

136. $a_n = \frac{(-1)^n x^{n+1}}{n+1}$

137. $a_n = \frac{(-1)^n x^{2n}}{(2n)!}$

138. $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

139. $a_n = \frac{(-1)^n x^n}{n!}$

140. $a_n = \frac{(-1)^n x^{n+1}}{(n+1)!}$

141. $a_n = \frac{(-1)^{n+1}(x+1)^n}{n!}$

142. $a_n = \frac{(-1)^n(x-1)^n}{(n+1)!}$

Writing Partial Sums In Exercises 143–146, write the first five terms of the sequence. Then find an expression for the n th partial sum.

143. $a_n = \frac{1}{2n} - \frac{1}{2n+2}$

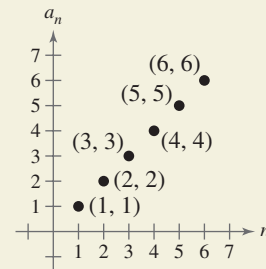
144. $a_n = \frac{1}{n} - \frac{1}{n+1}$

145. $a_n = \frac{1}{n+1} - \frac{1}{n+2}$

146. $a_n = \frac{1}{n} - \frac{1}{n+2}$

147. **Think About It** Does every finite series whose terms are integers have a finite sum? Explain.

148. **CAPSTONE** The graph represents the first six terms of a sequence.



- (a) Write the first six terms of the sequence.
- (b) Write an expression for the apparent n th term a_n of the sequence.
- (c) Use sigma notation to represent the partial sum of the first 50 terms of the sequence.

Cumulative Mixed Review

Operations with Matrices In Exercises 149–152, find, if possible, (a) $A - B$, (b) $2B - 3A$, (c) AB , and (d) BA .

149. $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 6 & -3 \end{bmatrix}$

150. $A = \begin{bmatrix} 10 & 7 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -12 \\ 8 & 11 \end{bmatrix}$

151. $A = \begin{bmatrix} -2 & -3 & 6 \\ 4 & 5 & 7 \\ 1 & 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 6 \\ 0 & 3 & 1 \end{bmatrix}$

152. $A = \begin{bmatrix} -1 & 4 & 0 \\ 5 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

8.2 Arithmetic Sequences and Partial Sums

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence**.

Definition of Arithmetic Sequence

A sequence is **arithmetic** when the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

is arithmetic when there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \dots = d.$$

The number d is the **common difference** of the sequence.

What you should learn

- Recognize, write, and find the n th terms of arithmetic sequences.
- Find n th partial sums of arithmetic sequences.
- Use arithmetic sequences to model and solve real-life problems.

Why you should learn it

Arithmetic sequences can reduce the amount of time it takes to find the sum of a sequence of numbers with a common difference. In Exercise 83 on page 587, you will use an arithmetic sequence to find the number of bricks needed to lay a brick patio.

Example 1 Examples of Arithmetic Sequences

a. The sequence whose n th term is

$$4n + 3$$

is arithmetic. The common difference between consecutive terms is 4.

$$7, 11, 15, 19, \dots, 4n + 3, \dots \quad \text{Begin with } n = 1.$$

$$\underbrace{11 - 7 = 4}$$

b. The sequence whose n th term is

$$7 - 5n$$

is arithmetic. The common difference between consecutive terms is -5 .

$$2, -3, -8, -13, \dots, 7 - 5n, \dots \quad \text{Begin with } n = 1.$$

$$\underbrace{-3 - 2 = -5}$$

c. The sequence whose n th term is

$$\frac{1}{4}(n + 3)$$

is arithmetic. The common difference between consecutive terms is $\frac{1}{4}$.

$$1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \dots, \frac{n + 3}{4}, \dots \quad \text{Begin with } n = 1.$$

$$\underbrace{\frac{5}{4} - 1 = \frac{1}{4}}$$

CHECKPOINT Now try Exercise 11.

The sequence $1, 4, 9, 16, \dots$, whose n th term is n^2 , is *not* arithmetic. The difference between the first two terms is

$$a_2 - a_1 = 4 - 1 = 3$$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5.$$

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The n th term of an arithmetic sequence can be derived from the pattern below.

$a_1 = a_1$ 1st term

$a_2 = a_1 + d$ 2nd term

$a_3 = a_1 + 2d$ 3rd term

$a_4 = a_1 + 3d$ 4th term

$a_5 = a_1 + 4d$ 5th term



$a_n = a_1 + (n - 1)d$ n th term



The result is summarized in the following definition.

The n th Term of an Arithmetic Sequence

The n th term of an arithmetic sequence has the form

$$a_n = a_1 + (n - 1)d$$

where d is the common difference between consecutive terms of the sequence and a_1 is the first term of the sequence.

Example 2 Finding the n th Term of an Arithmetic Sequence

Find a formula for the n th term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

Solution

You know that the formula for the n th term is of the form $a_n = a_1 + (n - 1)d$. Moreover, because the common difference is $d = 3$ and the first term is $a_1 = 2$, the formula must have the form

$a_n = 2 + 3(n - 1)$ *Substitute 2 for a_1 and 3 for d .*

or $a_n = 3n - 1$. The sequence therefore has the following form.

$2, 5, 8, 11, 14, \dots, 3n - 1, \dots$

A graph of the first 15 terms of the sequence is shown in Figure 8.6. Notice that the points lie on a line. This makes sense because a_n is a linear function of n . In other words, the terms “arithmetic” and “linear” are closely connected.

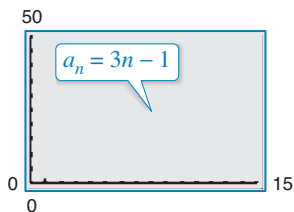


Figure 8.6

CHECKPOINT Now try Exercise 21.

Explore the Concept



Consider the following sequences.

$1, 4, 7, 10, \dots, 3n - 2, \dots$

$-5, 1, 7, 13, \dots, 6n - 11, \dots$

$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \dots, \frac{7}{2} - n, \dots$

What relationship do you observe between successive terms of these sequences?

Technology Tip



You can use a graphing utility to generate the arithmetic sequence

in Example 2 by using the following steps.

2 **ENTER**

3 **+** **ANS**

Now press the *enter* key repeatedly to generate the terms of the sequence.

Example 3 Writing the Terms of an Arithmetic Sequence

The fourth term of an arithmetic sequence is 20, and the 13th term is 65. Write the first several terms of this sequence.

Solution

You know that $a_4 = 20$ and $a_{13} = 65$. So, you must add the common difference d nine times to the fourth term to obtain the 13th term. Therefore, the fourth and 13th terms of the sequence are related by

$$a_{13} = a_4 + 9d. \quad a_4 \text{ and } a_{13} \text{ are nine terms apart.}$$

Using $a_4 = 20$ and $a_{13} = 65$, you have

$$65 = 20 + 9d.$$

So, you can conclude that $d = 5$, which implies that the sequence is as follows.

$$\begin{array}{cccccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & \dots \\ 5, & 10, & 15, & 20, & 25, & 30, & 35, & 40, & 45, & 50, & 55, & 60, & 65, & \dots \end{array}$$

 **CHECKPOINT** Now try Exercise 35.

When you know the n th term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$ th term by using the *recursion formula*

$$a_{n+1} = a_n + d. \quad \text{Recursion formula}$$

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the preceding term. For instance, when you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

Example 4 Using a Recursion Formula

Find the ninth term of the arithmetic sequence whose first two terms are 2 and 9.

Solution

You know that the sequence is arithmetic. Also, $a_1 = 2$ and $a_2 = 9$. So, the common difference for this sequence is

$$d = 9 - 2 = 7.$$

There are two ways to find the ninth term. One way is simply to write out the first nine terms (by repeatedly adding 7).

$$2, 9, 16, 23, 30, 37, 44, 51, 58$$

Another way to find the ninth term is to first find a formula for the n th term. Because the common difference is $d = 7$ and the first term is $a_1 = 2$, the formula must have the form

$$a_n = 2 + 7(n - 1). \quad \text{Substitute 2 for } a_1 \text{ and 7 for } d.$$

Therefore, a formula for the n th term is

$$a_n = 7n - 5$$

which implies that the ninth term is

$$a_9 = 7(9) - 5 = 58.$$

 **CHECKPOINT** Now try Exercise 43.

Technology Tip



Most graphing utilities have a built-in function that will display the terms of an arithmetic sequence. For instructions on how to use the *sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

```
seq(2+7(n-1), n, 1
, 9)
30 37 44 51 58)
```



The Sum of a Finite Arithmetic Sequence

There is a simple formula for the *sum* of a finite arithmetic sequence.

The Sum of a Finite Arithmetic Sequence (See the proof on page 633.)

The sum of a finite arithmetic sequence with n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Be sure you see that this formula works only for *arithmetic* sequences.

Example 5 Finding the Sum of a Finite Arithmetic Sequence

Find the sum: $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

Solution

To begin, notice that the sequence is arithmetic (with a common difference of 2). Moreover, the sequence has 10 terms. So, the sum of the sequence is

$$\begin{aligned} S_n &= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 \\ &= \frac{n}{2}(a_1 + a_n) && \text{Formula for sum of an arithmetic sequence} \\ &= \frac{10}{2}(1 + 19) && \text{Substitute 10 for } n, 1 \text{ for } a_1, \text{ and } 19 \text{ for } a_n. \\ &= 5(20) = 100. && \text{Simplify.} \end{aligned}$$

 **CHECKPOINT** Now try Exercise 57.

The sum of the first n terms of an infinite sequence is called the **n th partial sum**. The n th partial sum of an arithmetic sequence can be found by using the formula for the sum of a finite arithmetic sequence.

Example 6 Finding a Partial Sum of an Arithmetic Sequence

Find the 150th partial sum of the arithmetic sequence $5, 16, 27, 38, 49, \dots$

Solution

For this arithmetic sequence, you have $a_1 = 5$ and $d = 16 - 5 = 11$. So,

$$a_n = 5 + 11(n - 1)$$

and the n th term is $a_n = 11n - 6$. Therefore, $a_{150} = 11(150) - 6 = 1644$, and the sum of the first 150 terms is

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) && \text{\textit{n}th partial sum formula} \\ &= \frac{150}{2}(5 + 1644) && \text{Substitute 150 for } n, 5 \text{ for } a_1, \text{ and } 1644 \text{ for } a_n. \\ &= 75(1649) && \text{Simplify.} \\ &= 123,675. && \text{\textit{n}th partial sum} \end{aligned}$$

 **CHECKPOINT** Now try Exercise 65.

Applications

Example 7 Total Sales



A small business sells \$20,000 worth of sports memorabilia during its first year. The owner of the business has set a goal of increasing annual sales by \$15,000 each year for 19 years. Assuming that this goal is met, find the total sales during the first 20 years this business is in operation.

Algebraic Solution

The annual sales form an arithmetic sequence in which $a_1 = 20,000$ and $d = 15,000$. So,

$$a_n = 20,000 + 15,000(n - 1)$$

and the n th term of the sequence is

$$a_n = 15,000n + 5000.$$

This implies that the 20th term of the sequence is

$$\begin{aligned} a_{20} &= 15,000(20) + 5000 \\ &= 300,000 + 5000 \\ &= 305,000. \end{aligned}$$

The sum of the first 20 terms of the sequence is

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) && \textit{nth partial sum formula} \\ &= \frac{20}{2}(20,000 + 305,000) && \textit{Substitute 20 for } n, \\ & && \textit{20,000 for } a_1, \textit{ and} \\ & && \textit{305,000 for } a_n. \\ &= 10(325,000) && \textit{Simplify.} \\ &= 3,250,000. && \textit{Simplify.} \end{aligned}$$

So, the total sales for the first 20 years are \$3,250,000.

CHECKPOINT Now try Exercise 85.

Numerical Solution

The annual sales form an arithmetic sequence in which $a_1 = 20,000$ and $d = 15,000$. So,

$$a_n = 20,000 + 15,000(n - 1)$$

and the n th term of the sequence is

$$a_n = 15,000n + 5000.$$

Use the *list editor* of a graphing utility to create a table that shows the sales for each of the first 20 years and the total sales for the first 20 years, as shown in Figure 8.7.

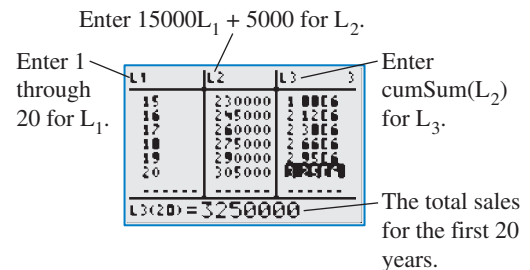


Figure 8.7

So, the total sales for the first 20 years are \$3,250,000.

Figure 8.8 shows the annual sales for the business in Example 7. Notice that the annual sales for the business follow a *linear growth* pattern. In other words, saying that a quantity increases arithmetically is the same as saying that it increases linearly.

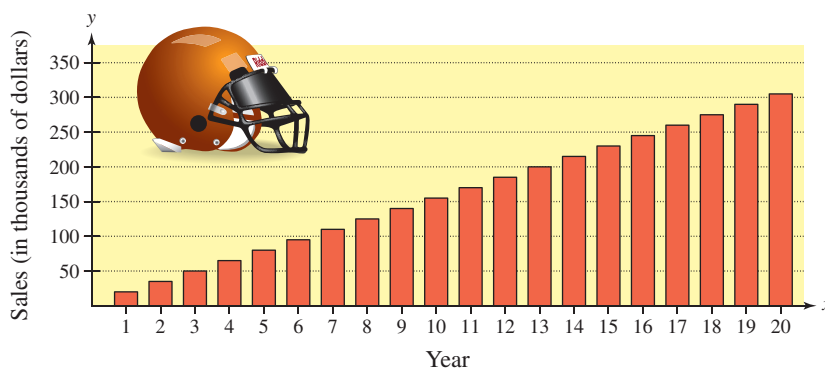


Figure 8.8

If you go on to take a course in calculus, you will study sequences and series in detail. You will learn that sequences and series play a major role in the study of calculus.

8.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

- The n th term of an arithmetic sequence has the form _____.
- The formula $S_n = \frac{n}{2}(a_1 + a_n)$ can be used to find the sum of the first n terms of an arithmetic sequence, called the _____.
- How do you know when a sequence is arithmetic?
- Is 4 or 1 the common difference of the arithmetic sequence $a_n = 4n + 1$?

Procedures and Problem Solving

Determining Whether a Sequence Is Arithmetic In Exercises 5–10, determine whether or not the sequence is arithmetic. If it is, find the common difference.

- 10, 8, 6, 4, 2, . . .
- $3, \frac{5}{2}, 2, \frac{3}{2}, 1, . . .$
- 3.7, 4.3, 4.9, 5.5, 6.1, . . .
- 4, 9, 14, 19, 24, . . .
- $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, . . .$
- $1^2, 2^2, 3^2, 4^2, 5^2, . . .$

Writing the Terms of a Sequence In Exercises 11–20, write the first five terms of the sequence. Determine whether or not the sequence is arithmetic. If it is, find the common difference. (Assume n begins with 1.)

- $a_n = 8 + 13n$
- $a_n = \frac{1}{n+1}$
- $a_n = 150 - 7n$
- $a_n = 3 + 2(-1)^n$
- $a_n = (-1)^n$
- $a_n = 2^n + n$
- $a_n = 1 + (n - 1)4$
- $a_n = 2^{n-1}$
- $a_n = 3 - 4(n + 6)$
- $a_n = (-1)^{2n+1}$

Finding the n th Term of an Arithmetic Sequence In Exercises 21–30, find a formula for a_n for the arithmetic sequence.

- $a_1 = 1, d = 3$
- $a_1 = 100, d = -8$
- $4, \frac{3}{2}, -1, -\frac{7}{2}, . . .$
- $a_1 = 5, a_4 = 15$
- $a_3 = 94, a_6 = 85$
- $a_1 = 15, d = 4$
- $a_1 = 0, d = -\frac{2}{3}$
- 10, 5, 0, -5, -10, . . .
- $a_1 = -4, a_5 = 16$
- $a_5 = 190, a_{10} = 115$

Writing the Terms of an Arithmetic Sequence In Exercises 31–38, write the first five terms of the arithmetic sequence. Use the *table* feature of a graphing utility to verify your results.

- $a_1 = 5, d = 6$
- $a_1 = -10, d = -12$
- $a_8 = 26, a_{12} = 42$
- $a_3 = 19, a_{15} = -1.7$
- $a_1 = 5, d = -\frac{3}{4}$
- $a_4 = 16, a_{10} = 46$
- $a_6 = -38, a_{11} = -73$
- $a_5 = 16, a_{14} = 38.5$

Writing Terms of an Arithmetic Sequence In Exercises 39–42, write the first five terms of the arithmetic sequence. Find the common difference and write the n th term of the sequence as a function of n .

- $a_1 = 15, a_{k+1} = a_k + 4$
- $a_1 = 6, a_{k+1} = a_k + 5$
- $a_1 = \frac{3}{5}, a_{k+1} = -\frac{1}{10} + a_k$
- $a_1 = 1.5, a_{k+1} = a_k - 2.5$

Using a Recursion Formula In Exercises 43–46, the first two terms of the arithmetic sequence are given. Find the missing term. Use the *table* feature of a graphing utility to verify your results.

- $a_1 = 5, a_2 = 11, a_{10} = \square$
- $a_1 = 3, a_2 = 13, a_9 = \square$
- $a_1 = 4.2, a_2 = 6.6, a_7 = \square$
- $a_1 = -0.7, a_2 = -13.8, a_8 = \square$

Graphing Terms of a Sequence In Exercises 47–50, use a graphing utility to graph the first 10 terms of the sequence. (Assume n begins with 1.)

- $a_n = 15 - \frac{3}{2}n$
- $a_n = 0.4n - 2$
- $a_n = -5 + 2n$
- $a_n = -1.3n + 7$

Finding Terms of a Sequence In Exercises 51–56, use the *table* feature of a graphing utility to find the first 10 terms of the sequence. (Assume n begins with 1.)

- $a_n = 4n - 5$
- $a_n = 20 - \frac{3}{4}n$
- $a_n = 1.5 + 0.05n$
- $a_n = 17 + 3n$
- $a_n = \frac{4}{5}n + 12$
- $a_n = 8 - 12.5n$

Finding the Sum of a Finite Arithmetic Sequence In Exercises 57–64, find the sum of the finite arithmetic sequence.

- $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$
- $1 + 4 + 7 + 10 + 13 + 16 + 19$

59. $-1 + (-3) + (-5) + (-7) + (-9)$
 60. $-5 + (-3) + (-1) + 1 + 3 + 5$
 61. Sum of the first 100 positive integers
 62. Sum of the first 50 negative integers
 63. Sum of the integers from -100 to 30
 64. Sum of the integers from -10 to 50

Finding a Partial Sum of an Arithmetic Sequence In Exercises 65–70, find the indicated n th partial sum of the arithmetic sequence.

- ✓ 65. $8, 20, 32, 44, \dots, n = 10$
 66. $2, 8, 14, 20, \dots, n = 25$
 67. $0.5, 1.3, 2.1, 2.9, \dots, n = 10$
 68. $4.2, 3.7, 3.2, 2.7, \dots, n = 12$
 69. $a_1 = 100, a_{25} = 220, n = 25$
 70. $a_1 = 15, a_{100} = 307, n = 100$

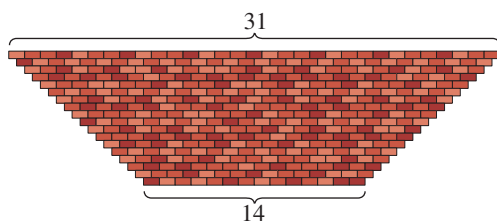
Finding a Partial Sum of an Arithmetic Sequence In Exercises 71–76, find the partial sum without using a graphing utility.

71. $\sum_{n=1}^{50} n$ 72. $\sum_{n=1}^{100} 2n$
 73. $\sum_{n=11}^{30} n - \sum_{n=1}^{10} n$ 74. $\sum_{n=51}^{100} n - \sum_{n=1}^{50} n$
 75. $\sum_{n=1}^{500} (n + 8)$ 76. $\sum_{n=1}^{100} \frac{8 - 3n}{16}$

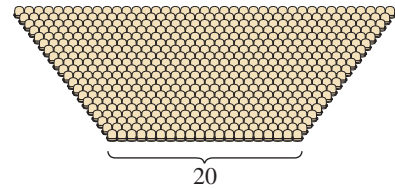
Finding a Partial Sum Using a Graphing Utility In Exercises 77–82, use a graphing utility to find the partial sum.

77. $\sum_{n=1}^{20} (2n + 1)$ 78. $\sum_{n=0}^{50} (50 - 2n)$
 79. $\sum_{n=1}^{100} \frac{n + 1}{2}$ 80. $\sum_{n=0}^{100} \frac{4 - n}{4}$
 81. $\sum_{i=1}^{60} (250 - \frac{2}{5}i)$ 82. $\sum_{j=1}^{200} (10.5 + 0.025j)$

83. **Why you should learn it** (p. 581) A brick patio has the approximate shape of a trapezoid, as shown in the figure. The patio has 18 rows of bricks. The first row has 14 bricks and the 18th row has 31 bricks. How many bricks are in the patio?




84. **Performing Arts** An auditorium has 20 rows of seats. There are 20 seats in the first row, 21 seats in the second row, 22 seats in the third row, and so on. How many seats are there in all 20 rows?



- ✓ 85. **Sales** A hardware store makes a profit of \$30,000 during its first year. The store owner sets a goal of increasing profits by \$5000 each year for 4 years. Assuming that this goal is met, find the total profit during the first 5 years of business.
86. **Physics** An object with negligible air resistance is dropped from a plane. During the first second of fall, the object falls 16 feet; during the second second, it falls 48 feet; during the third second, it falls 80 feet; and during the fourth second, it falls 112 feet. Assume this pattern continues. How many feet will the object fall in 8 seconds?

87. MODELING DATA


The table shows the sales a_n (in billions of dollars) for Coca-Cola Enterprises from 2001 through 2008. (Source: Coca-Cola Enterprises, Inc.)

 Year	Sales, a_n (in billions of dollars)
2001	15.7
2002	16.9
2003	17.3
2004	18.2
2005	18.7
2006	19.8
2007	20.9
2008	21.8

- (a) Use the *regression* feature of a graphing utility to find an arithmetic sequence for the data. Let n represent the year, with $n = 1$ corresponding to 2001.
- (b) Use the sequence from part (a) to approximate the annual sales for Coca-Cola Enterprises for the years 2001 through 2008. How well does the model fit the data?
- (c) Use the sequence to find the total sales for Coca-Cola Enterprises over the period from 2001 through 2008.
- (d) Use the sequence to predict the total sales over the period from 2009 through 2016. Is your total reasonable? Explain.

88. MODELING DATA

The table shows the numbers a_n (in thousands) of master's degrees conferred in the United States from 2000 through 2007. (Source: U.S. National Center for Education Statistics)

 Year	Master's degrees conferred, a_n (in thousands)
2000	457
2001	468
2002	482
2003	513
2004	559
2005	575
2006	594
2007	605

- (a) Use the *regression* feature of a graphing utility to find an arithmetic sequence for the data. Let n represent the year, with $n = 0$ corresponding to 2000.
- (b) Use the sequence from part (a) to approximate the numbers of master's degrees conferred for the years 2000 through 2007. How well does the model fit the data?
- (c) Use the sequence to find the total number of master's degrees conferred over the period from 2000 through 2007.
- (d) Use the sequence to predict the total number of master's degrees conferred over the period from 2008 through 2018. Is your total reasonable? Explain.

Conclusions

True or False? In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.

- 89. Given the n th term and the common difference of an arithmetic sequence, it is possible to find the $(n + 1)$ th term.
- 90. If the only known information about a finite arithmetic sequence is its first term and its last term, then it is possible to find the sum of the sequence.

Finding Terms of a Sequence In Exercises 91 and 92, find the first 10 terms of the sequence.

- 91. $a_1 = x, d = 2x$ 92. $a_1 = -y, d = 5y$
- 93. **Think About It** The sum of the first 20 terms of an arithmetic sequence with a common difference of 3 is 650. Find the first term.
- 94. **Writing** Explain how to use the first two terms of an arithmetic sequence to find the n th term.

- 95. **Think About It** The sum of the first n terms of an arithmetic sequence with first term a_1 and common difference d is S_n . Determine the sum when each term is increased by 5. Explain.
- 96. **Think About It** In each sequence, decide whether it is possible to fill in the blanks to form an arithmetic sequence. If so, find a recursion formula for the sequence. Explain how you found your answers.
 - (a) $-7, \square, \square, \square, \square, \square, 11$
 - (b) $17, \square, \square, \square, \square, \square, \square, 59$
 - (c) $2, 6, \square, \square, 162$
 - (d) $4, 7.5, \square, \square, \square, \square, \square, 28.5$
 - (e) $8, 12, \square, \square, \square, 60.75$
- 97. **Writing** Carl Friedrich Gauss, a famous nineteenth century mathematician, was a child prodigy. It was said that when Gauss was 10 he was asked by his teacher to add the numbers from 1 to 100. Almost immediately, Gauss found the answer by mentally finding the summation. Write an explanation of how he arrived at his conclusion, and then find the formula for the sum of the first n natural numbers.

98. CAPSTONE Describe the characteristics of an arithmetic sequence. Give an example of a sequence that is arithmetic and one that is not arithmetic.

Finding the Sum of a Finite Arithmetic Sequence In Exercises 99–102, find the sum using the method from Exercise 97.

- 99. The first 200 natural numbers
- 100. The first 100 even natural numbers from 2 to 200, inclusive
- 101. The first 51 odd natural numbers from 1 to 101, inclusive
- 102. The first 100 multiples of 4 from 4 to 400, inclusive

Cumulative Mixed Review

Gauss-Jordan Elimination In Exercises 103 and 104, use Gauss-Jordan elimination to solve the system of equations.

- 103.
$$\begin{cases} 2x - y + 7z = -10 \\ 3x + 2y - 4z = 17 \\ 6x - 5y + z = -20 \end{cases}$$
- 104.
$$\begin{cases} -x + 4y + 10z = 4 \\ 5x - 3y + z = 31 \\ 8x + 2y - 3z = -5 \end{cases}$$

- 105. **Make a Decision** To work an extended application analyzing the amount of municipal waste recovered in the United States from 1983 through 2008, visit this textbook's *Companion Website*. (Data Source: U.S. Environmental Protection Agency)

8.3 Geometric Sequences and Series

Geometric Sequences

In Section 8.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence**. Consecutive terms of a geometric sequence have a common *ratio*.

Definition of Geometric Sequence

A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \dots, a_n, \dots$ is geometric when there is a number r such that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \dots = r, \quad r \neq 0.$$

The number r is the **common ratio** of the sequence.

Example 1 Examples of Geometric Sequences

- a. The sequence whose n th term is 2^n is geometric. For this sequence, the common ratio between consecutive terms is 2.

$$2, 4, 8, 16, \dots, 2^n, \dots$$

Begin with $n = 1$.

$$\frac{4}{2} = 2$$

- b. The sequence whose n th term is $4(3^n)$ is geometric. For this sequence, the common ratio between consecutive terms is 3.

$$12, 36, 108, 324, \dots, 4(3^n), \dots$$

Begin with $n = 1$.

$$\frac{36}{12} = 3$$

- c. The sequence whose n th term is $(-\frac{1}{3})^n$ is geometric. For this sequence, the common ratio between consecutive terms is $-\frac{1}{3}$.

$$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots, \left(-\frac{1}{3}\right)^n, \dots$$

Begin with $n = 1$.

$$\frac{1/9}{-1/3} = -\frac{1}{3}$$

CHECKPOINT Now try Exercise 7.

The sequence

$$1, 4, 9, 16, \dots$$

whose n th term is n^2 , is *not* geometric. The ratio of the second term to the first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is

$$\frac{a_3}{a_2} = \frac{9}{4}$$

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What you should learn

- Recognize, write, and find the n th terms of geometric sequences.
- Find n th partial sums of geometric sequences.
- Find sums of infinite geometric series.
- Use geometric sequences to model and solve real-life problems.

Why you should learn it

Geometric sequences can reduce the amount of time it takes to find the sum of a sequence of numbers with a common ratio. For instance, Exercise 109 on page 598 shows how to use a geometric sequence to find the total vertical distance traveled by a bouncing ball.



Study Tip



In Example 1, notice that each of the geometric sequences has an n th term of the form ar^n , where r is the common ratio of the sequence.

The n th Term of a Geometric Sequence

The n th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$a_1, a_2, a_3, a_4, a_5, \dots, a_n, \dots$$

$$a_1, a_1 r, a_1 r^2, a_1 r^3, a_1 r^4, \dots, a_1 r^{n-1}, \dots$$

When you know the n th term of a geometric sequence, you can find the $(n + 1)$ th term by multiplying by r . That is,

$$a_{n+1} = a_n r.$$

Example 2 Finding the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is

$$a_1 = 3$$

and whose common ratio is

$$r = 2.$$

Solution

Starting with 3, repeatedly multiply by 2 to obtain the following.

$$a_1 = 3 \quad \text{1st term}$$

$$a_2 = 3(2^1) = 6 \quad \text{2nd term}$$

$$a_3 = 3(2^2) = 12 \quad \text{3rd term}$$

$$a_4 = 3(2^3) = 24 \quad \text{4th term}$$

$$a_5 = 3(2^4) = 48 \quad \text{5th term}$$

CHECKPOINT Now try Exercise 17.

Example 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

Algebraic Solution

$$a_n = a_1 r^{n-1} \quad \text{Formula for a geometric sequence}$$

$$a_{15} = 20(1.05)^{15-1} \quad \text{Substitute 20 for } a_1, \text{ 1.05 for } r, \text{ and 15 for } n.$$

$$\approx 39.60 \quad \text{Use a calculator.}$$

CHECKPOINT Now try Exercise 31.

Numerical Solution

For this sequence, $r = 1.05$ and $a_1 = 20$. So, $a_n = 20(1.05)^{n-1}$. Use the *table* feature of a graphing utility to create a table that shows the terms of the sequence, as shown in Figure 8.9.

n	$u(n)$
9	29.549
10	31.027
11	32.578
12	34.207
13	35.917
14	37.711
15	39.599

$u(n) = 39.59863199$

The number in the 15th row is the 15th term of the sequence.

So, $a_{15} \approx 39.60$.

Figure 8.9

Technology Tip



You can use a graphing utility to generate the geometric sequence in Example 2 by using the following steps.

3 **ENTER**

2 **⊗** **ANS**

Now press the *enter* key repeatedly to generate the terms of the sequence.

Example 4 Finding a Term of a Geometric Sequence

Find a formula for the n th term of the following geometric sequence. What is the ninth term of the sequence?

$$5, 15, 45, \dots$$

Solution

The common ratio of this sequence is

$$r = \frac{15}{5} = 3.$$

Because the first term is $a_1 = 5$, the formula must have the form

$$a_n = a_1 r^{n-1} = 5(3)^{n-1}.$$

You can determine the ninth term ($n = 9$) to be

$$\begin{aligned} a_9 &= 5(3)^{9-1} && \text{Substitute 9 for } n. \\ &= 5(6561) && \text{Use a calculator.} \\ &= 32,805. && \text{Simplify.} \end{aligned}$$

A graph of the first nine terms of the sequence is shown in Figure 8.10. Notice that the points lie on an exponential curve. This makes sense because a_n is an exponential function of n .

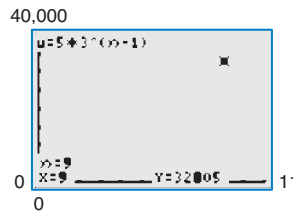


Figure 8.10

CHECKPOINT Now try Exercise 39.

When you know *any* two terms of a geometric sequence, you can use that information to find a formula for the n th term of the sequence.

Example 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is

$$\frac{125}{64}.$$

Find the 14th term. (Assume that the terms of the sequence are positive.)

Solution

The 10th term is related to the fourth term by the equation

$$a_{10} = a_4 r^6. \quad \text{Multiply 4th term by } r^{10-4}.$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\begin{aligned} \frac{125}{64} &= 125r^6 \\ \frac{1}{64} &= r^6 \quad \Rightarrow \quad \frac{1}{2} = r \end{aligned}$$

You can obtain the 14th term by multiplying the 10th term by r^4 .

$$a_{14} = a_{10} r^4 = \frac{125}{64} \left(\frac{1}{2}\right)^4 = \frac{125}{1024}$$

CHECKPOINT Now try Exercise 45.

Technology Tip

Most graphing utilities have a built-in function that will display the terms of a geometric sequence. For instructions on how to use the *sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

```
seq(5*3^(n-1),n,
1,9)
45 10935 32805)
```

Study Tip

Remember that r is the common ratio of consecutive terms of a geometric sequence. So, in Example 5,

$$\begin{aligned} a_{10} &= a_1 r^9 \\ &= a_1 \cdot r \cdot r \cdot r \cdot r \cdot r^6 \\ &= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6 \end{aligned}$$

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence (See the proof on page 633.)

The sum of the finite geometric sequence

$$a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \dots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^n a_1r^{i-1} = a_1 \left(\frac{1-r^n}{1-r} \right).$$

Example 6 Finding the Sum of a Finite Geometric Sequence

Find the following sum.

$$\sum_{n=1}^{12} 4(0.3)^n$$

Solution

By writing out a few terms, you have

$$\sum_{n=1}^{12} 4(0.3)^n = 4(0.3)^1 + 4(0.3)^2 + 4(0.3)^3 + \dots + 4(0.3)^{12}.$$

Now, because

$$a_1 = 4(0.3), \quad r = 0.3, \quad \text{and} \quad n = 12$$

you can apply the formula for the sum of a finite geometric sequence to obtain

$$\begin{aligned} \sum_{n=1}^{12} 4(0.3)^n &= a_1 \left(\frac{1-r^n}{1-r} \right) && \text{Formula for sum of a finite} \\ & && \text{geometric sequence} \\ &= 4(0.3) \left[\frac{1-(0.3)^{12}}{1-0.3} \right] && \text{Substitute } 4(0.3) \text{ for } a_1, 0.3 \text{ for } r, \\ & && \text{and } 12 \text{ for } n. \\ &\approx 1.71. && \text{Use a calculator.} \end{aligned}$$

CHECKPOINT Now try Exercise 55.

When using the formula for the sum of a geometric sequence, be careful to check that the index begins at $i = 1$. For an index that begins at $i = 0$, you must adjust the formula for the n th partial sum. For instance, if the index in Example 6 had begun with $n = 0$, then the sum would have been

$$\begin{aligned} \sum_{n=0}^{12} 4(0.3)^n &= 4(0.3)^0 + \sum_{n=1}^{12} 4(0.3)^n \\ &= 4 + \sum_{n=1}^{12} 4(0.3)^n \\ &\approx 4 + 1.71 \\ &= 5.71. \end{aligned}$$

Technology Tip



Using the *sum* *sequence* feature of a graphing utility, you can calculate the sum of the sequence in Example 6 to be about 1.7142848, as shown below.

```
sum(seq(4+0.3^n,
n,1,12))
1.714284803
```

Calculate the sum beginning at $n = 0$. You should obtain a sum of about 5.7142848.



What's Wrong?

You use a graphing utility to find the sum

$$\sum_{n=1}^6 7(0.5)^n$$

as shown in the figure. What's wrong?

```
?*(1-.5^6)/(1-.5)
13.78125
```


Geometric Series

The sum of the terms of an infinite geometric sequence is called an **infinite geometric series** or simply a **geometric series**.

The formula for the sum of a *finite geometric sequence* can, depending on the value of r , be extended to produce a formula for the sum of an *infinite geometric series*. Specifically, if the common ratio r has the property that

$$|r| < 1$$

then it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1 \left(\frac{1 - r^n}{1 - r} \right) \rightarrow a_1 \left(\frac{1 - 0}{1 - r} \right) \quad \text{as } n \rightarrow \infty.$$

This result is summarized as follows.

The Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + \cdots$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1 r^i = \frac{a_1}{1 - r}.$$

Note that when

$$|r| \geq 1$$

the series does not have a sum.

Example 7 Finding the Sum of an Infinite Geometric Series

Find each sum.

a. $\sum_{n=0}^{\infty} 4(0.6)^n$

b. $3 + 0.3 + 0.03 + 0.003 + \cdots$

Solution

$$\begin{aligned} \text{a. } \sum_{n=0}^{\infty} 4(0.6)^n &= 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^n + \cdots \\ &= \frac{4}{1 - 0.6} && \frac{a_1}{1 - r} \\ &= 10 \end{aligned}$$

$$\begin{aligned} \text{b. } 3 + 0.3 + 0.03 + 0.003 + \cdots &= 3 + 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots \\ &= \frac{3}{1 - 0.1} && \frac{a_1}{1 - r} \\ &= \frac{10}{3} \\ &\approx 3.33 \end{aligned}$$

 **CHECKPOINT** Now try Exercise 75.

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Explore the Concept

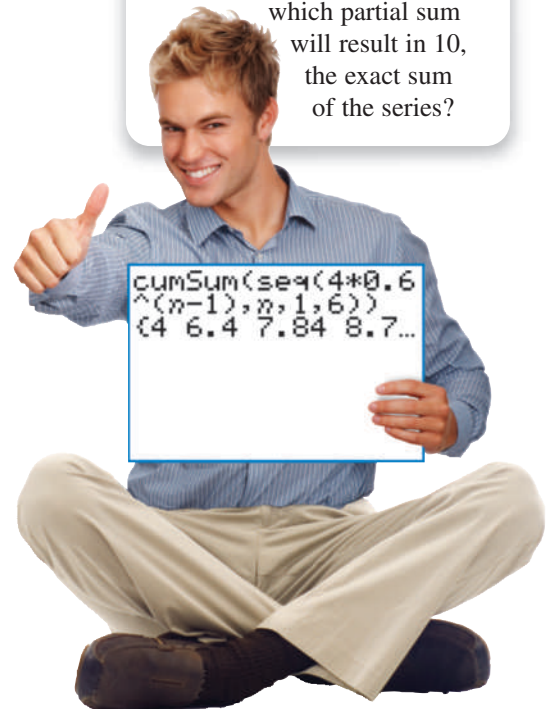


Notice that the formula for the sum of an infinite geometric series requires that $|r| < 1$. What happens when $r = 1$ or $r = -1$? Give examples of infinite geometric series for which $|r| > 1$ and convince yourself that they do not have finite sums.

Explore the Concept



On most graphing utilities, it is not possible to find the sum of the series in Example 7(a) because you cannot enter ∞ as the upper limit of summation. Can you still find the sum using a graphing utility without entering ∞ ? If so, which partial sum will result in 10, the exact sum of the series?



Application

Example 8 Increasing Annuity

A deposit of \$50 is made on the first day of each month in an account that pays 6% compounded monthly. What is the balance at the end of 2 years? (This type of savings plan is called an **increasing annuity**.)

Solution

Recall from Section 3.1 that the compound interest formula is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Formula for compound interest}$$

To find the balance in the account after 24 months, consider each of the 24 deposits separately. The first deposit will gain interest for 24 months, and its balance will be

$$\begin{aligned} A_{24} &= 50\left(1 + \frac{0.06}{12}\right)^{24} \\ &= 50(1.005)^{24}. \end{aligned}$$

The second deposit will gain interest for 23 months, and its balance will be

$$\begin{aligned} A_{23} &= 50\left(1 + \frac{0.06}{12}\right)^{23} \\ &= 50(1.005)^{23}. \end{aligned}$$

The last deposit will gain interest for only 1 month, and its balance will be

$$\begin{aligned} A_1 &= 50\left(1 + \frac{0.06}{12}\right)^1 \\ &= 50(1.005). \end{aligned}$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $a_1 = 50(1.005)$, $r = 1.005$, and $n = 24$, you have

$$\begin{aligned} S_n &= a_1\left(\frac{1 - r^n}{1 - r}\right) && \text{Formula for sum of a finite} \\ &&& \text{geometric sequence} \\ S_{24} &= 50(1.005)\left[\frac{1 - (1.005)^{24}}{1 - 1.005}\right] && \text{Substitute } 50(1.005) \text{ for } a_1, \\ &&& \text{1.005 for } r, \text{ and } 24 \text{ for } n. \\ &\approx \$1277.96. && \text{Simplify.} \end{aligned}$$

You can use the *sum sequence* feature of a graphing utility to verify this result, as shown in Figure 8.11.

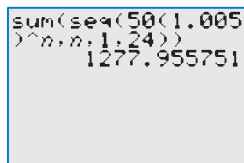


Figure 8.11

 **CHECKPOINT** Now try Exercise 101.



Personal Financial Advisor

8.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–3, fill in the blank(s).

1. A sequence is called a _____ sequence if the ratios of consecutive terms are the same. This ratio is called the _____ ratio.
2. The n th term of a geometric sequence has the form $a_n = \underline{\hspace{2cm}}$.
3. The sum of the terms of an infinite geometric sequence is called a _____.
4. Can a geometric sequence have a common ratio of 0?
5. For what values of the common ratio r is it possible to find the sum of an infinite geometric series?
6. Which formula represents the sum of a *finite geometric sequence*? an *infinite geometric series*?

$$(a) S = \frac{a_1}{1-r}, |r| < 1 \quad (b) S_n = a_1 \left(\frac{1-r^n}{1-r} \right)$$

Procedures and Problem Solving

Identifying a Geometric Sequence In Exercises 7–16, determine whether or not the sequence is geometric. If it is, find the common ratio.

- ✓ 7. 5, 15, 45, 135, . . . 8. 3, 12, 48, 192, . . .
 9. 6, 18, 30, 42, . . . 10. 1, -2, 4, -8, . . .
 11. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$ 12. 5, 1, 0.2, 0.04, . . .
 13. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \dots$ 14. 9, -6, 4, $-\frac{8}{3}, \dots$
 15. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ 16. $\frac{1}{5}, \frac{2}{7}, \frac{3}{9}, \frac{4}{11}, \dots$

Finding the Terms of a Geometric Sequence In Exercises 17–24, write the first five terms of the geometric sequence.

- ✓ 17. $a_1 = 6, r = 3$ 18. $a_1 = 4, r = 2$
 19. $a_1 = 1, r = \frac{1}{2}$ 20. $a_1 = 2, r = \frac{1}{3}$
 21. $a_1 = 5, r = -\frac{1}{10}$ 22. $a_1 = 6, r = -\frac{1}{4}$
 23. $a_1 = 1, r = e$ 24. $a_1 = 4, r = \sqrt{3}$

Finding the n th Term of a Geometric Sequence In Exercises 25–30, write the first five terms of the geometric sequence. Find the common ratio and write the n th term of the sequence as a function of n .

25. $a_1 = 64, a_{k+1} = \frac{1}{2}a_k$ 26. $a_1 = 81, a_{k+1} = \frac{1}{3}a_k$
 27. $a_1 = 9, a_{k+1} = 2a_k$ 28. $a_1 = 5, a_{k+1} = -3a_k$
 29. $a_1 = 6, a_{k+1} = -\frac{3}{2}a_k$ 30. $a_1 = 30, a_{k+1} = -\frac{2}{3}a_k$

Finding a Term of a Geometric Sequence In Exercises 31–38, find the indicated term of the geometric sequence (a) using the *table* feature of a graphing utility and (b) algebraically.

- ✓ 31. $a_1 = 6, r = \frac{1}{3}$, 12th term
 32. $a_1 = 24, r = \frac{3}{2}$, 8th term

33. $a_1 = 8, r = -\frac{4}{3}$, 7th term
 34. $a_1 = 8, r = -\frac{3}{4}$, 9th term
 35. $a_1 = -1, r = 3$, 6th term
 36. $a_1 = -\frac{1}{128}, r = 2$, 12th term
 37. $a_1 = 500, r = -1.02$, 14th term
 38. $a_1 = 1000, r = 1.005$, 11th term

Finding a Term of a Geometric Sequence In Exercises 39–42, find a formula for the n th term of the geometric sequence. Then find the indicated term of the geometric sequence.

- ✓ 39. 9th term: 7, 21, 63, . . .
 40. 7th term: 3, 36, 432, . . .
 41. 10th term: 5, 30, 180, . . .
 42. 22nd term: 4, 8, 16, . . .

Finding a Term of a Geometric Sequence In Exercises 43–46, find the indicated term of the geometric sequence.

43. $a_1 = 4, a_4 = \frac{1}{2}$, 10th term
 44. $a_1 = 5, a_3 = \frac{45}{4}$, 8th term
 ✓ 45. $a_2 = -18, a_5 = \frac{2}{3}$, 6th term
 46. $a_3 = \frac{16}{3}, a_5 = \frac{64}{27}$, 7th term

Using a Graphing Utility In Exercises 47–50, use a graphing utility to graph the first 10 terms of the sequence.

47. $a_n = 12(-0.75)^{n-1}$
 48. $a_n = 20(1.25)^{n-1}$
 49. $a_n = 2(1.3)^{n-1}$
 50. $a_n = 10(-1.2)^{n-1}$

Finding a Sequence of Partial Sums In Exercises 51 and 52, find the sequence of the first five partial sums $S_1, S_2, S_3, S_4,$ and S_5 of the geometric sequence by adding terms.

51. $8, -4, 2, -1, \frac{1}{2}, \dots$ 52. $8, 12, 18, 27, \frac{81}{2}, \dots$

Finding a Sequence of Partial Sums In Exercises 53 and 54, use a graphing utility to create a table showing the sequence of the first 10 partial sums $S_1, S_2, S_3, \dots,$ and S_{10} for the series.

53. $\sum_{n=1}^{\infty} 16\left(\frac{1}{2}\right)^{n-1}$ 54. $\sum_{n=1}^{\infty} 4(0.2)^{n-1}$

Finding the Sum of a Finite Geometric Sequence In Exercises 55–64, find the sum. Use a graphing utility to verify your result.

✓ 55. $\sum_{n=1}^9 2^{n-1}$ 56. $\sum_{n=1}^9 (-2)^{n-1}$

57. $\sum_{i=1}^7 64\left(-\frac{1}{2}\right)^{i-1}$ 58. $\sum_{i=1}^6 32\left(\frac{1}{4}\right)^{i-1}$

59. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^n$ 60. $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^n$

61. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$ 62. $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1}$

63. $\sum_{n=0}^5 300(1.06)^n$ 64. $\sum_{n=0}^6 500(1.04)^n$

Using Summation Notation In Exercises 65–68, use summation notation to write the sum.

65. $5 + 15 + 45 + \dots + 3645$

66. $7 + 14 + 28 + \dots + 896$

67. $2 - \frac{1}{2} + \frac{1}{8} - \dots + \frac{1}{2048}$

68. $15 - 3 + \frac{3}{5} - \dots - \frac{3}{625}$

Finding the Sum of an Infinite Geometric Series In Exercises 69–82, find the sum of the infinite geometric series, if possible. If not possible, explain why.

69. $\sum_{n=0}^{\infty} 10\left(\frac{4}{5}\right)^n$ 70. $\sum_{n=0}^{\infty} 6\left(\frac{2}{3}\right)^n$

71. $\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^n$ 72. $\sum_{n=0}^{\infty} 9\left(-\frac{2}{3}\right)^n$

73. $\sum_{n=1}^{\infty} 2\left(\frac{7}{3}\right)^{n-1}$ 74. $\sum_{n=1}^{\infty} 8\left(\frac{5}{3}\right)^{n-1}$

✓ 75. $\sum_{n=0}^{\infty} 10(0.11)^n$ 76. $\sum_{n=0}^{\infty} 5(0.45)^n$

77. $\sum_{n=0}^{\infty} -3(-0.9)^n$ 78. $\sum_{n=0}^{\infty} -10(-0.2)^n$

79. $9 + 6 + 4 + \frac{8}{3} + \dots$ 80. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \dots$

81. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \dots$

82. $-6 + 5 - \frac{25}{6} + \frac{125}{36} - \dots$

Writing a Repeating Decimal as a Rational Number In Exercises 83–86, find the rational number representation of the repeating decimal.

83. $0.\overline{36}$ 84. $0.\overline{297}$

85. $1.2\overline{5}$ 86. $1.3\overline{8}$

Identifying a Sequence In Exercises 87–94, determine whether the sequence associated with the series is arithmetic or geometric. Find the common difference or ratio and find the sum of the first 15 terms.

87. $8 + 16 + 32 + 64 + \dots$

88. $17 + 14 + 11 + 8 + \dots$

89. $90 + 30 + 10 + \frac{10}{3} + \dots$

90. $\frac{5}{4} + \frac{7}{4} + \frac{9}{4} + \frac{11}{4} + \dots$

91. $\sum_{n=1}^{\infty} 6n$ 92. $\sum_{n=1}^{\infty} 2^{n-1}$

93. $\sum_{n=0}^{\infty} 6(0.8)^n$ 94. $\sum_{n=0}^{\infty} \frac{n+4}{4}$

95. **Compound Interest** A principal of \$1000 is invested at 3% interest. Find the amount after 10 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

96. **Annuity** A deposit of \$100 is made at the beginning of each month in an account that pays 3% interest, compounded monthly. The balance A in the account at the end of 5 years is given by

$$A = 100\left(1 + \frac{0.03}{12}\right)^1 + \dots + 100\left(1 + \frac{0.03}{12}\right)^{60}.$$

Find A .

97. **Annuity** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r , compounded monthly. The balance A after t years is given by

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^2 + \dots + P\left(1 + \frac{r}{12}\right)^{12t}.$$

Show that the balance is given by

$$A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right).$$

98. **Annuity** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r , compounded continuously. The balance A after t years is given by $A = Pe^{r/12} + Pe^{2r/12} + \dots + Pe^{12tr/12}$. Show that the balance is given by

$$A = \frac{Pe^{r/12}(e^{rt} - 1)}{e^{r/12} - 1}.$$

Annuities In Exercises 99–102, consider making monthly deposits of P dollars in a savings account earning an annual interest rate r . Use the results of Exercises 97 and 98 to find the balances A after t years if the interest is compounded (a) monthly and (b) continuously.

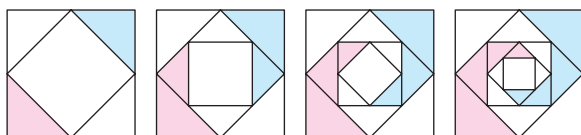
99. $P = \$50$, $r = 7\%$, $t = 20$ years

100. $P = \$75$, $r = 4\%$, $t = 25$ years

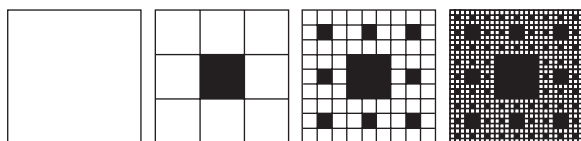
✓ 101. $P = \$100$, $r = 5\%$, $t = 40$ years

102. $P = \$20$, $r = 6\%$, $t = 50$ years

103. **Geometry** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the resulting triangles are shaded. After this process is repeated five more times, determine the total area of the shaded region.



104. **Geometry** The sides of a square are 27 inches in length. New squares are formed by dividing the original square into nine squares. The center square is then shaded (see figure). After this process is repeated three more times, determine the total area of the shaded region.

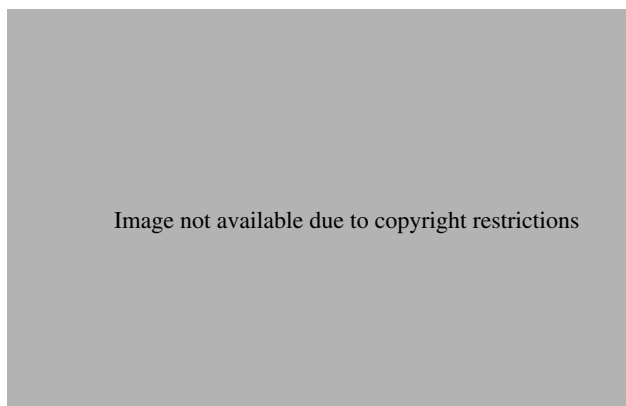


105. **Physics** The temperature of water in an ice cube tray is 70°F when it is placed in a freezer. Its temperature n hours after being placed in the freezer is 20% less than 1 hour earlier.

- Find a formula for the n th term of the geometric sequence that gives the temperature of the water n hours after it is placed in the freezer.
- Find the temperatures of the water 6 hours and 12 hours after it is placed in the freezer.
- Use a graphing utility to graph the sequence to approximate the time required for the water to freeze.

106. **Fractals** In a *fractal*, a geometric figure is repeated at smaller and smaller scales. The sphereflake shown is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. Attached to the large sphere are nine spheres of radius $\frac{1}{3}$. Attached to each of the smaller spheres are nine spheres of radius $\frac{1}{9}$. This process is continued infinitely.


Eric Haines



- Write a formula in series notation that gives the surface area of the sphereflake.
- Write a formula in series notation that gives the volume of the sphereflake.
- Is the surface area of the sphereflake finite or infinite? Is the volume finite or infinite? If either is finite, find the value.

107. MODELING DATA

The table shows the mid-year populations a_n of China (in millions) from 2002 through 2008. (Source: U.S. Census Bureau)

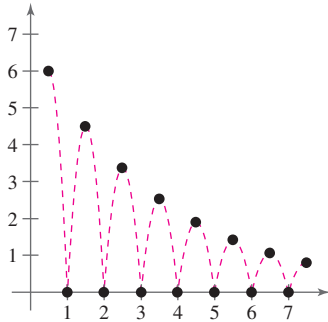
 Year	Population, a_n
2002	1284.3
2003	1291.5
2004	1298.8
2005	1306.3
2006	1314.0
2007	1321.9
2008	1330.0

- Use the *exponential regression* feature of a graphing utility to find a geometric sequence that models the data. Let n represent the year, with $n = 2$ corresponding to 2002.
- Use the sequence from part (a) to describe the rate at which the population of China is growing.
- Use the sequence from part (a) to predict the population of China in 2015. The U.S. Census Bureau predicts the population of China will be 1393.4 million in 2015. How does this value compare with your prediction?
- Use the sequence from part (a) to determine when the population of China will reach 1.35 billion.

108. Manufacturing The manufacturer of a new food processor plans to produce and sell 8000 units per year. Each year, 10% of all units sold become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.9(8000)]$ units will be in use after 2 years, and so on.

- (a) Write a formula in series notation for the number of units that will be operative after n years.
- (b) Find the numbers of units that will be operative after 10 years, 15 years, and 20 years.
- (c) If this trend continues indefinitely, will the number of units that will be operative be finite? If so, how many? If not, explain your reasoning.

109. Why you should learn it (p. 589) A ball is dropped from a height of 6 feet and begins bouncing, as shown in the figure. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance the ball travels before coming to rest.



Conclusions

True or False? In Exercises 110–112, determine whether the statement is true or false. Justify your answer.

- 110. For a geometric sequence in which the quotient of term a_7 and term a_5 is 16, the common ratio is 8.
- 111. The first n terms of a geometric sequence with a common ratio of 1 are the same as the first n terms of an arithmetic sequence with a common difference of 0 if both sequences have the same first term.
- 112. You can find the n th term of a geometric sequence by multiplying its common ratio by the first term of the sequence raised to the $(n - 1)$ th power.

Finding the Terms of a Geometric Sequence In Exercises 113 and 114, write the first five terms of the geometric sequence.

- 113. $a_1 = 3, r = \frac{x}{2}$
- 114. $a_1 = \frac{1}{2}, r = 7x$

Finding a Term of a Geometric Sequence In Exercises 115 and 116, find the indicated term of the geometric sequence.

- 115. $a_1 = 100, r = e^x$, 9th term
- 116. $a_1 = 4, r = \frac{4x}{3}$, 6th term

117. Exploration Use a graphing utility to graph each function. Identify the horizontal asymptote of the graph and determine its relationship to the sum.

- (a) $f(x) = 6 \left[\frac{1 - (0.5)^x}{1 - (0.5)} \right], \sum_{n=0}^{\infty} 6 \left(\frac{1}{2} \right)^n$
- (b) $f(x) = 2 \left[\frac{1 - (0.8)^x}{1 - (0.8)} \right], \sum_{n=0}^{\infty} 2 \left(\frac{4}{5} \right)^n$

- 118. **Writing** Write a brief paragraph explaining why the terms of a geometric sequence decrease in magnitude when $-1 < r < 1$.
- 119. **Writing** Write a brief paragraph explaining how to use the first two terms of a geometric sequence to find the n th term.

120. CAPSTONE The terms of a geometric sequence can be written as

$$a_1, a_2 = a_1r, a_3 = a_2r, a_4 = a_3r, \dots$$

Write each term of the sequence in terms of a_1 and r . Then, based on the pattern, write the n th term of the geometric sequence. What do you notice?

Cumulative Mixed Review

- 121. **Average Speed** A truck traveled at an average speed of 50 miles per hour on a 200-mile trip. On the return trip, the average speed was 42 miles per hour. Find the average speed for the round trip.
- 122. **Work Rate** Your friend can mow a lawn in 4 hours and you can mow it in 6 hours. How long will it take both of you to mow the lawn working together?

Finding a Determinant In Exercises 123 and 124, find the determinant of the matrix.

123. $\begin{bmatrix} -1 & 3 & 4 \\ -2 & 8 & 0 \\ 2 & 5 & -1 \end{bmatrix}$ 124. $\begin{bmatrix} -1 & 0 & 4 \\ -4 & 3 & 5 \\ 0 & 2 & -3 \end{bmatrix}$

125. Make a Decision To work an extended application analyzing the monthly profits of a clothing manufacturer over a period of 36 months, visit this textbook's *Companion Website*.

8.4 The Binomial Theorem

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that provides a quick method of raising a binomial to a power. To begin, look at the expansion of

$$(x + y)^n$$

for several values of n .

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetric roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The sum of the powers of each term is n . For instance, in the expansion of

$$(x + y)^5$$

the sum of the powers of each term is 5.

$$(x + y)^5 = x^5 + \overbrace{5x^4y^1}^{4+1=5} + \overbrace{10x^3y^2}^{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients**. To find them, you can use the **Binomial Theorem**.

What you should learn

- Use the Binomial Theorem to calculate binomial coefficients.
- Use binomial coefficients to write binomial expansions.
- Use Pascal's Triangle to calculate binomial coefficients.

Why you should learn it

You can use binomial coefficients to model and solve real-life problems. For instance, in Exercise 118 on page 606, you will use binomial coefficients to write the expansion of a model that represents the amount of child support collected in the United States.



The Binomial Theorem (See the proof on page 634.)

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_n C_r x^{n-r} y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r} y^r$ is

$${}_n C_r = \frac{n!}{(n-r)!r!}$$

The symbol

$$\binom{n}{r}$$

is often used in place of ${}_n C_r$ to denote binomial coefficients.

Example 1 Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_8C_2$ b. $\binom{10}{3}$

c. ${}_7C_0$ d. $\binom{8}{8}$

Solution

a. ${}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$

b. $\binom{10}{3} = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

c. ${}_7C_0 = \frac{7!}{7! \cdot 0!} = 1$

d. $\binom{8}{8} = \frac{8!}{0! \cdot 8!} = 1$

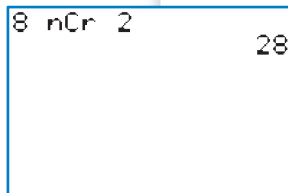
CHECKPOINT Now try Exercise 5.

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) of Example 1, there is a simple pattern for evaluating binomial coefficients that works because there will always be factorial terms that divide out from the expression.

$${}_8C_2 = \frac{\overbrace{8 \cdot 7}^{2 \text{ factors}}}{\underbrace{2 \cdot 1}_{2 \text{ factorial}}} \quad \text{and} \quad \binom{10}{3} = \frac{\overbrace{10 \cdot 9 \cdot 8}^{3 \text{ factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{3 \text{ factorial}}}$$

Technology Tip

Most graphing utilities are programmed to evaluate ${}_nC_r$. For instructions on how to use the ${}_nC_r$ feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

**Example 2** Finding Binomial Coefficients

Find each binomial coefficient using the pattern shown above.

a. ${}_7C_3$ b. ${}_7C_4$ c. ${}_{12}C_1$ d. ${}_{12}C_{11}$

Solution

a. ${}_7C_3 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

b. ${}_7C_4 = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35$

c. ${}_{12}C_1 = \frac{12}{1} = 12$

d. ${}_{12}C_{11} = \frac{12!}{1! \cdot 11!} = \frac{(12) \cdot \cancel{11!}}{1! \cdot \cancel{11!}} = \frac{12}{1} = 12$

CHECKPOINT Now try Exercise 7.

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, it is true that

$${}_nC_r = {}_nC_{n-r}$$

Yuri Arcurs 2010/used under license from Shutterstock.com

Explore the Concept

Find each pair of binomial coefficients.

a. ${}_7C_0, {}_7C_7$ d. ${}_7C_1, {}_7C_6$

b. ${}_8C_0, {}_8C_8$ e. ${}_8C_1, {}_8C_7$

c. ${}_{10}C_0, {}_{10}C_{10}$ f. ${}_{10}C_1, {}_{10}C_9$

What do you observe about the pairs in (a), (b), and (c)? What do you observe about the pairs in (d), (e), and (f)? Write two conjectures from your observations. Develop a convincing argument for your two conjectures.

Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial**. The formulas for binomial coefficients give you an easy way to expand binomials, as demonstrated in the next four examples.

Example 3 Expanding a Binomial

Write the expansion of the expression $(x + 1)^3$.

Solution

The binomial coefficients are

$${}_3C_0 = 1, {}_3C_1 = 3, {}_3C_2 = 3, \text{ and } {}_3C_3 = 1.$$

Therefore, the expansion is as follows.

$$\begin{aligned}(x + 1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3) \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

 **CHECKPOINT** Now try Exercise 23.

To expand binomials representing *differences*, rather than sums, you alternate signs, as shown in the next example.

Example 4 Expanding a Binomial

Write the expansion of the expression $(x - 1)^3$.

Solution

Expand using the binomial coefficient from Example 3.

$$\begin{aligned}(x - 1)^3 &= [x + (-1)]^3 \\ &= (1)x^3 + (3)x^2(-1) + (3)x(-1)^2 + (1)(-1)^3 \\ &= x^3 - 3x^2 + 3x - 1\end{aligned}$$

 **CHECKPOINT** Now try Exercise 27.

Example 5 Expanding Binomial Expressions

Write the expansion of each expression.

a. $(2x - 3)^4$ b. $(x - 2y)^4$

Solution

The binomial coefficients are

$${}_4C_0 = 1, {}_4C_1 = 4, {}_4C_2 = 6, {}_4C_3 = 4, \text{ and } {}_4C_4 = 1.$$

Therefore, the expansions are as follows.

$$\begin{aligned}\text{a. } (2x - 3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4) \\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

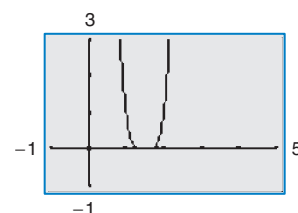
$$\begin{aligned}\text{b. } (x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4 \\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

 **CHECKPOINT** Now try Exercise 35.

Technology Tip



You can use a graphing utility to check the expansion in Example 5(a) by graphing the original binomial expression and the expansion in the same viewing window. The graphs should coincide, as shown below.



Example 6 Expanding a Binomial

Write the expansion of the expression

$$(x^2 + 4)^3.$$

Solution

Expand using the binomial coefficients from Example 3.

$$\begin{aligned}(x^2 + 4)^3 &= (1)(x^2)^3 + (3)(x^2)^2(4) + (3)x^2(4^2) + (1)(4^3) \\ &= x^6 + 12x^4 + 48x^2 + 64\end{aligned}$$

 Now try Exercise 37.

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$ th term is

$${}_n C_r x^{n-r} y^r.$$

For instance, to find the third term of the expression in Example 6, you could use the formula above with $n = 3$ and $r = 2$ to obtain

$$\begin{aligned}{}_3 C_2 (x^2)^{3-2} \cdot 4^2 &= 3(x^2) \cdot 16 \\ &= 48x^2.\end{aligned}$$

Example 7 Finding a Term or Coefficient in a Binomial Expansion

- Find the sixth term of $(a + 2b)^8$.
- Find the coefficient of the term $a^6 b^5$ in the expansion of $(2a - 5b)^{11}$.

Solution

- Because the formula is for the $(r + 1)$ th term, r is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$ and $y = 2b$.

$$\begin{aligned}{}_n C_r x^{n-r} y^r &= {}_8 C_5 a^{8-5} (2b)^5 \\ &= 56 \cdot a^3 \cdot (2b)^5 \\ &= 56(2^5) a^3 b^5 \\ &= 1792 a^3 b^5\end{aligned}$$

- Note that

$$(2a - 5b)^{11} = [2a + (-5b)]^{11}.$$


So,

$$n = 11, \quad r = 5, \quad x = 2a, \quad \text{and} \quad y = -5b.$$

Substitute these values to obtain

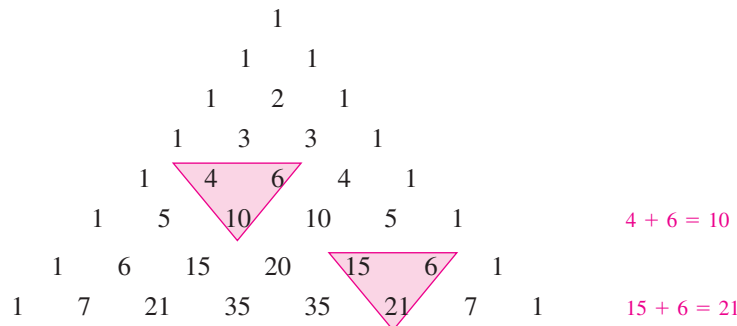
$$\begin{aligned}{}_n C_r x^{n-r} y^r &= {}_{11} C_5 (2a)^6 (-5b)^5 \\ &= (462)(64a^6)(-3125b^5) \\ &= -92,400,000 a^6 b^5.\end{aligned}$$

So, the coefficient is $-92,400,000$.

 Now try Exercises 55 and 67.

Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle**. This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).



The first and last number in each row of Pascal's Triangle is 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions, as follows.

$$\begin{array}{ll}
 (x + y)^0 = 1 & \text{0th row} \\
 (x + y)^1 = 1x + 1y & \text{1st row} \\
 (x + y)^2 = 1x^2 + 2xy + 1y^2 & \text{2nd row} \\
 (x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 & \text{3rd row} \\
 (x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 & \vdots \\
 (x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5 \\
 (x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6 \\
 (x + y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7
 \end{array}$$

The top row of Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion

$$(x + y)^0 = 1.$$

Similarly, the next row is called the *first row* because it corresponds to the binomial expansion

$$(x + y)^1 = 1(x) + 1(y).$$

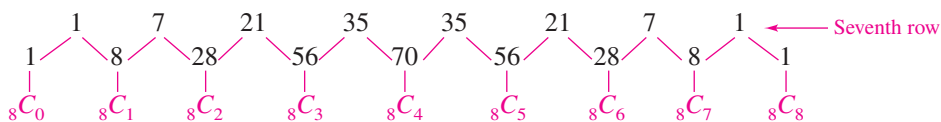
In general, the *n*th row of Pascal's Triangle gives the coefficients of $(x + y)^n$.

Example 8 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0 \quad {}_8C_1 \quad {}_8C_2 \quad {}_8C_3 \quad {}_8C_4 \quad {}_8C_5 \quad {}_8C_6 \quad {}_8C_7 \quad {}_8C_8$$

Solution



CHECKPOINT Now try Exercise 71.

Explore the Concept



Complete the table and describe the result.

n	r	${}_nC_r$	${}_nC_{n-r}$
9	5	<input type="text"/>	<input type="text"/>
7	1	<input type="text"/>	<input type="text"/>
12	4	<input type="text"/>	<input type="text"/>
6	0	<input type="text"/>	<input type="text"/>
10	7	<input type="text"/>	<input type="text"/>

What characteristics of Pascal's Triangle are illustrated by this table?

8.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

- The notation used to denote a binomial coefficient is _____ or _____.
- When you write out the coefficients for a binomial that is raised to a power, you are _____ a _____.
- List two ways to find binomial coefficients.
- In the expression of $(x + y)^3$, what is the sum of the powers of the third term?

Procedures and Problem Solving

Finding Binomial Coefficients In Exercises 5–16, find the binomial coefficient.

- ${}_{7}C_5$
- ${}_{20}C_{15}$
- ${}_{14}C_1$
- $\binom{12}{0}$
- $\binom{10}{4}$
- $\binom{100}{98}$
- ${}_{8}C_6$
- ${}_{12}C_3$
- ${}_{18}C_{17}$
- $\binom{20}{20}$
- $\binom{10}{6}$
- $\binom{100}{2}$

Finding Binomial Coefficients In Exercises 17–22, use a graphing utility to find ${}_nC_r$.

- ${}_{41}C_{36}$
- ${}_{34}C_4$
- ${}_{50}C_{48}$
- ${}_{500}C_{498}$
- ${}_{250}C_2$
- ${}_{1000}C_2$

Expanding a Binomial In Exercises 23–54, use the Binomial Theorem to expand and simplify the expression.

- $(x + 2)^4$
- $(a + 3)^3$
- $(y - 4)^3$
- $(x + y)^5$
- $(r + 3s)^6$
- $(x - y)^5$
- $(1 - 4x)^3$
- $(x^2 + 2)^4$
- $(x^2 - 5)^5$
- $(x^2 + y^2)^4$
- $(x^3 - y)^6$
- $\left(\frac{1}{x} + y\right)^5$
- $\left(\frac{2}{x} - y\right)^4$
- $(x + 1)^6$
- $(a + 2)^4$
- $(y - 2)^5$
- $(x + y)^6$
- $(4x + 3y)^4$
- $(2x - y)^5$
- $(5 - 2y)^3$
- $(3 - y^2)^3$
- $(y^2 + 1)^6$
- $(x^2 + y^2)^6$
- $(2x^3 - y)^5$
- $\left(\frac{1}{x} + 2y\right)^6$
- $\left(\frac{2}{x} - 3y\right)^5$

- $(4x - 1)^3 - 2(4x - 1)^4$
- $(x + 3)^5 - 4(x + 3)^4$
- $2(x - 3)^4 + 5(x - 3)^2$
- $3(x + 1)^5 + 4(x + 1)^3$
- $-3(x - 2)^3 - 4(x + 1)^6$
- $5(x + 2)^5 - 2(x - 1)^2$

Finding a Term in a Binomial Expansion In Exercises 55–62, find the specified n th term in the expansion of the binomial.

- $(x + 8)^{10}$, $n = 4$
- $(x - 5)^6$, $n = 7$
- $(x - 6y)^5$, $n = 3$
- $(x - 10z)^7$, $n = 4$
- $(4x + 3y)^9$, $n = 8$
- $(5a + 6b)^5$, $n = 5$
- $(10x - 3y)^{12}$, $n = 10$
- $(7x + 2y)^{15}$, $n = 7$

Finding a Coefficient in a Binomial Expansion In Exercises 63–70, find the coefficient a of the given term in the expansion of the binomial.

- | Binomial | Term |
|----------------------|-----------|
| 63. $(x + 3)^{12}$ | ax^5 |
| 64. $(x + 4)^{12}$ | ax^4 |
| 65. $(x - 2y)^{10}$ | ax^8y^2 |
| 66. $(4x - y)^{10}$ | ax^2y^8 |
| 67. $(3x - 2y)^9$ | ax^6y^3 |
| 68. $(2x - 3y)^8$ | ax^4y^4 |
| 69. $(x^2 + y)^{10}$ | ax^8y^6 |
| 70. $(z^2 - 1)^{12}$ | az^6 |

Using Pascal's Triangle In Exercises 71–74, use Pascal's Triangle to find the binomial coefficient.

- ${}_{7}C_4$
- ${}_{6}C_3$
- ${}_{6}C_5$
- ${}_{5}C_2$

Using Pascal's Triangle In Exercises 75–80, expand the binomial by using Pascal's Triangle to determine the coefficients.

75. $(3t - 2v)^4$ 76. $(5v - 2z)^4$
 77. $(2x - 3y)^5$ 78. $(5y + 2)^5$
 79. $(x + 2y)^5$ 80. $(2v + 3)^6$

Using the Binomial Theorem In Exercises 81–84, use the Binomial Theorem to expand and simplify the expression.

81. $(\sqrt{x} + 5)^3$ 82. $(2\sqrt{t} - 1)^3$
 83. $(x^{2/3} - y^{1/3})^3$ 84. $(u^{3/5} + v^{1/5})^5$

Expanding an Expression In Exercises 85–90, expand the expression in the difference quotient and simplify.

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$

85. $f(x) = x^3$ 86. $f(x) = x^4$
 87. $f(x) = x^6$ 88. $f(x) = x^8$
 89. $f(x) = \sqrt{x}$ 90. $f(x) = \frac{1}{x}$

Expanding a Complex Number In Exercises 91–104, use the Binomial Theorem to expand the complex number. Simplify your result. (Remember that $i = \sqrt{-1}$.)

91. $(1 + i)^4$ 92. $(4 - i)^5$
 93. $(4 + i)^4$ 94. $(2 - i)^5$
 95. $(2 - 3i)^6$ 96. $(3 - 2i)^6$
 97. $(5 + \sqrt{-16})^3$ 98. $(5 + \sqrt{-9})^3$
 99. $(4 + \sqrt{3}i)^4$ 100. $(5 - \sqrt{3}i)^4$
 101. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$ 102. $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$
 103. $\left(\frac{1}{4} - \frac{\sqrt{3}}{4}i\right)^3$ 104. $\left(\frac{1}{3} - \frac{\sqrt{3}}{3}i\right)^3$

Using the Binomial Theorem to Approximate In Exercises 105–108, use the Binomial Theorem to approximate the quantity accurate to three decimal places. For example, in Exercise 105, use the expansion

$$(1.02)^8 = (1 + 0.02)^8 \\ = 1 + 8(0.02) + 28(0.02)^2 + \dots$$

105. $(1.02)^8$ 106. $(2.005)^{10}$
 107. $(2.99)^{12}$ 108. $(1.98)^9$

Using the Binomial Theorem In Exercises 109 and 110, use a graphing utility to graph f and g in the same viewing window. What is the relationship between the two graphs? Use the Binomial Theorem to write the polynomial function g in standard form.

109. $f(x) = x^3 - 4x$, $g(x) = f(x + 4)$

110. $f(x) = -x^4 + 4x^2 - 1$, $g(x) = f(x - 3)$

Using a Graphing Utility In Exercises 111 and 112, use a graphing utility to graph the functions in the given order and in the same viewing window. Compare the graphs. Which two functions have identical graphs, and why?

111. (a) $f(x) = (1 - x)^3$
 (b) $g(x) = 1 - 3x$
 (c) $h(x) = 1 - 3x + 3x^2$
 (d) $p(x) = 1 - 3x + 3x^2 - x^3$
 112. (a) $f(x) = \left(1 - \frac{1}{2}x\right)^4$
 (b) $g(x) = 1 - 2x + \frac{3}{2}x^2$
 (c) $h(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3$
 (d) $p(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

Finding a Probability In Exercises 113–116, consider n independent trials of an experiment in which each trial has two possible outcomes, success or failure. The probability of a success on each trial is p and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.

113. A fair coin is tossed seven times. To find the probability of obtaining four heads, evaluate the term

$${}_7C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^7$.

114. The probability of a baseball player getting a hit during any given time at bat is $\frac{1}{4}$. To find the probability that the player gets three hits during the next 10 times at bat, evaluate the term

$${}_{10}C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^7$$

in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

115. The probability of a sales representative making a sale to any one customer is $\frac{1}{3}$. The sales representative makes eight contacts a day. To find the probability of making four sales, evaluate the term

$${}_8C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^4$$

in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^8$.

116. To find the probability that the sales representative in Exercise 115 makes four sales when the probability of a sale to any one customer is $\frac{1}{2}$, evaluate the term

$${}_8C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^4$$

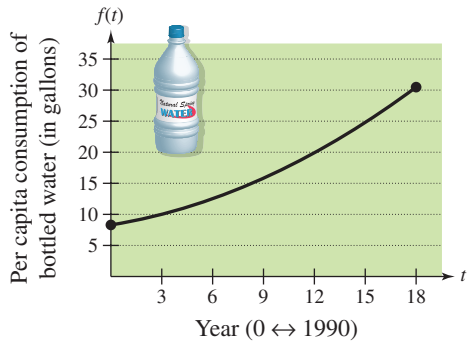
in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

117. MODELING DATA

The per capita consumption of bottled water f (in gallons) in the United States from 1990 through 2008 can be approximated by the model

$$f(t) = 0.044t^2 + 0.44t + 8.3, \quad 0 \leq t \leq 18$$

where t represents the year, with $t = 0$ corresponding to 1990 (see figure). (Source: Economic Research Service, U.S. Department of Agriculture)



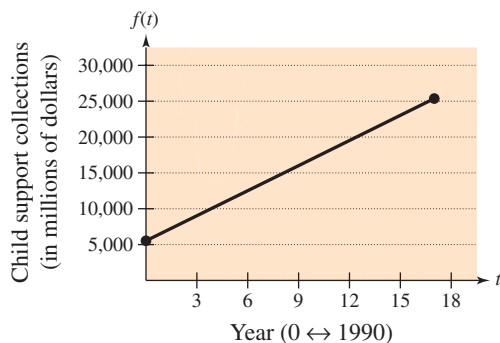
- (a) Adjust the model so that $t = 0$ corresponds to 2000 rather than 1990. To do this, shift the graph of f 10 units to the left and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.
- (b) Use a graphing utility to graph f and g in the same viewing window.

118. Why you should learn it (p. 599) The amounts f (in millions of dollars) of child support collected in the United States from 1990 through 2007 can be approximated by the model



$$f(t) = 0.26t^2 + 1162.5t + 5531, \quad 0 \leq t \leq 17$$

where t represents the year, with $t = 0$ corresponding to 1990 (see figure). (Source: U.S. Department of Health and Human Services)



- (a) Adjust the model so that $t = 0$ corresponds to 2000 rather than 1990. To do this, shift the graph of f 10 units to the left and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

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- (b) Use a graphing utility to graph f and g in the same viewing window.

Conclusions

True or False? In Exercises 119–122, determine whether the statement is true or false. Justify your answer.

- 119. The Binomial Theorem can be used to produce each row of Pascal’s Triangle.
- 120. A binomial that represents a difference cannot always be accurately expanded using the Binomial Theorem.
- 121. One of the terms in the expansion of $(x - 2y)^{12}$ is $7920x^4y^8$.
- 122. The x^{10} -term and the x^{14} -term in the expansion of $(x^2 + 3)^{12}$ have identical coefficients.
- 123. **Think About It** How many terms are in the expansion of $(x + y)^n$?
- 124. **Writing** In your own words, explain how to form the rows of Pascal’s Triangle. Then form rows 8–10 of Pascal’s Triangle.
- 125. **Error Analysis** You are a math instructor and receive the following solutions from one of your students on a quiz. Find the error(s) in each solution and write a short paragraph discussing ways that your student could avoid the error(s) in the future.

- (a) Find the second term in the expansion of $(2x - 3y)^5$.

$$\cancel{5(2x)^4(3y)^2 = 720x^4y^2}$$

- (b) Find the fourth term in the expansion of $(\frac{1}{2}x + 7y)^6$.

$$\cancel{{}_6C_4(\frac{1}{2}x)^2(7y)^4 = 9003.75x^2y^4}$$

126. CAPSTONE How do the expansions of $(x + y)^n$ and $(x - y)^n$ differ? Support your explanation with an example.

Proof In Exercises 127–130, prove the property for all integers r and n , where $0 \leq r \leq n$.

- 127. ${}_nC_r = {}_nC_{n-r}$
- 128. ${}_nC_0 - {}_nC_1 + {}_nC_2 - \dots \pm {}_nC_n = 0$
- 129. ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$
- 130. The sum of the numbers in the n th row of Pascal’s Triangle is 2^n .

Cumulative Mixed Review

Finding the Inverse of a Matrix In Exercises 131 and 132, find the inverse of the matrix.

131. $\begin{bmatrix} -1 & -4 \\ 1 & 2 \end{bmatrix}$ 132. $\begin{bmatrix} 11 & -12 \\ 2 & -2 \end{bmatrix}$

8.5 Counting Principles

Simple Counting Problems

The last two sections of this chapter present a brief introduction to some of the basic counting principles and their application to probability. In the next section, you will see that much of probability has to do with counting the number of ways an event can occur. The following two examples describe simple counting problems.

Example 1 Selecting Pairs of Numbers at Random



Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *returned to the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. In how many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the number of different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number



Second number



From this list, you can see that a sum of 12 can occur in five different ways.

CHECKPOINT Now try Exercise 9.

Example 2 Selecting Pairs of Numbers at Random



Eight pieces of paper are numbered from 1 to 8 and placed in a box. Two pieces of paper are drawn from the box *at the same time*, and the numbers on the pieces of paper are written down and totaled. In how many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the number of different ways that a sum of 12 can be obtained using two *different* numbers from 1 to 8.

First number



Second number



So, a sum of 12 can be obtained in four different ways.

CHECKPOINT Now try Exercise 11.

The difference between the counting problems in Examples 1 and 2 can be described by saying that the random selection in Example 1 occurs **with replacement**, whereas the random selection in Example 2 occurs **without replacement**, which eliminates the possibility of choosing two 6's.

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What you should learn

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve more complicated counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

Why you should learn it

You can use counting principles to solve counting problems that occur in real life. For instance, in Exercise 65 on page 615, you are asked to use counting principles to determine in how many ways a player can select six numbers in a Powerball lottery.



The Fundamental Counting Principle

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is always the best way to solve a counting problem. However, some events can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle**.

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1 , E_2 , and E_3 can occur is

$$m_1 \cdot m_2 \cdot m_3.$$

Example 3 Using the Fundamental Counting Principle



How many different pairs of letters from the English alphabet are possible?

Solution

There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is

$$26 \cdot 26 = 676.$$

CHECKPOINT Now try Exercise 13.

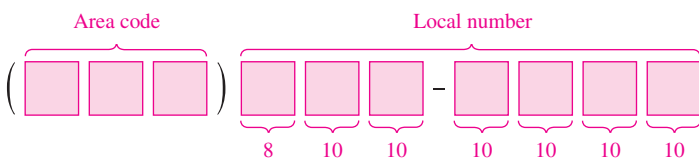
Example 4 Using the Fundamental Counting Principle



Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number*. How many different telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

Solution

Because the first digit cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of local telephone numbers that are possible within each area code is

$$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8,000,000.$$

CHECKPOINT Now try Exercise 19.

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 Finding the Number of Permutations of n Elements

How many permutations of the following letters are possible?

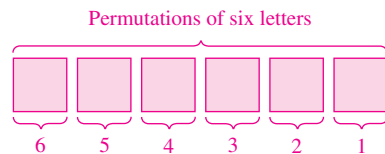
A B C D E F

Solution

Consider the following reasoning.

First position:	Any of the <i>six</i> letters
Second position:	Any of the remaining <i>five</i> letters
Third position:	Any of the remaining <i>four</i> letters
Fourth position:	Any of the remaining <i>three</i> letters
Fifth position:	Either of the remaining <i>two</i> letters
Sixth position:	The <i>one</i> remaining letter

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

CHECKPOINT Now try Exercise 37.

Number of Permutations of n Elements

The number of permutations of n elements is given by

$$n \cdot (n - 1) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose and order r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time**.

Example 6 Counting Horse Race Finishes 

Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

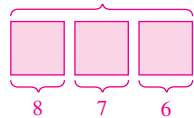
Solution

Here are the different possibilities.

Win (first position):	Eight choices
Place (second position):	Seven choices
Show (third position):	Six choices

The numbers of choices for the three positions are as follows.

Different orders of horses



Jockey

So, using the Fundamental Counting Principle, you can determine that there are

$$8 \cdot 7 \cdot 6 = 336$$

different ways in which the eight horses can come in first, second, and third.

 **CHECKPOINT** Now try Exercise 41.

Permutations of n Elements Taken r at a Time

The number of **permutations of n elements taken r at a time** is given by

$$\begin{aligned} {}_n P_r &= \frac{n!}{(n-r)!} \\ &= n(n-1)(n-2) \cdots (n-r+1). \end{aligned}$$

Using this formula, you can rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8 P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

Technology Tip

Most graphing utilities are programmed to evaluate ${}_n P_r$. Figure 8.12 shows how one graphing utility evaluates the permutation ${}_8 P_3$. For instructions on how to use the ${}_n P_r$ feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

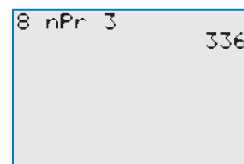


Figure 8.12

Remember that for permutations, order is important. So, to find the possible permutations of the letters

A, B, C, and D

taken three at a time, the permutations (A, B, D) and (B, A, D) would be counted as different because the *order* of the elements is different.

Consider, however, the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters would be

$${}_4P_4 = 4!.$$

However, not all of these arrangements would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdots + n_k.$$

The number of **distinguishable permutations** of the n objects is given by

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

Solution

This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways in which the letters can be written is

$$\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 2!} = 60.$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN
AABNAN	AABNNA	AANABN	AANANB
AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN
ABNANA	ABNNAA	ANAABN	ANAANB
ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB
ANNABA	ANNBAA	BAAANN	BAANAN
BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA
NAAABN	NAAANB	NAABAN	NAABNA
NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA
NBAAAN	NBAANA	NBANAA	NBNAAA
NNAAAB	NNAABA	NNABAA	NNBAAA

 **CHECKPOINT** Now try Exercise 49.

Combinations

When you count the number of possible permutations of a set of elements, order is important. As a final topic in this section, you will look at a method for selecting subsets of a larger set in which order *is not* important. Such subsets are called **combinations of n elements taken r at a time**. For instance, the combinations

$$\{A, B, C\} \quad \text{and} \quad \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. A common example of a combination is a card game in which the player is free to reorder the cards after they have been dealt.

Combinations of n Elements Taken r at a Time

The number of combinations of n elements taken r at a time is given by

$${}_n C_r = \frac{n!}{(n-r)!r!}$$

Note that the formula for ${}_n C_r$ is the same one given for binomial coefficients.

Example 8 Combinations of n Elements Taken r at a Time

- In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)
- A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt, the player may reorder them, so order is not important.)

Solution

- You can find the number of different ways in which the letters can be chosen by using the formula for the number of combinations of five elements taken three at a time, as follows.

$${}_5 C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \overset{3}{\cancel{3}}!}{2 \cdot 1 \cdot 3!} = 10$$

- You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$${}_{52} C_5 = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \overset{5}{\cancel{47}}!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \overset{5}{\cancel{47}}!} = 2,598,960$$

 **CHECKPOINT** Now try Exercise 61.

When solving problems involving counting principles, you need to be able to distinguish among the various counting principles in order to determine which is necessary to solve the problem correctly. To do this, ask yourself the following questions.

- Is the order of the elements important? *Permutation*
- Are the chosen elements a subset of a larger set in which order is not important? *Combination*
- Are there two or more separate events? *Fundamental Counting Principle*

Study Tip



In Example 8(a), you could also make a list of the different combinations of three letters chosen from five letters.

{A, B, C}	{A, B, D}
{A, B, E}	{A, C, D}
{A, C, E}	{A, D, E}
{B, C, D}	{B, C, E}
{B, D, E}	{C, D, E}

From this list, you can conclude that there are 10 different ways in which three letters can be chosen from five letters, which is the same answer obtained in Example 8(a).

8.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

- The _____ states that if there are m_1 ways for one event to occur and m_2 ways for a second event to occur, then there are $m_1 \cdot m_2$ ways for both events to occur.
- The number of _____ of n objects is given by $\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \dots \cdot n_k!}$.
- Is the ordering of n elements called a *permutation* or a *combination* of the elements?
- What do n and r represent in the formula ${}_nP_r = \frac{n!}{(n-r)!}$?

Procedures and Problem Solving

Selecting Numbers at Random In Exercises 5–12, determine the number of ways in which a computer can randomly generate one or more such integers, or pairs of integers, from 1 through 15.

- An odd integer
- An even integer
- A prime integer
- An integer that is greater than 6
- ✓ A pair of integers whose sum is 20
- An integer that is divisible by 4
- ✓ A pair of distinct integers whose sum is 20
- An integer that is divisible by 3
- ✓ **13. Audio Engineering** A customer can choose one of four amplifiers, one of six compact disc players, and one of five speaker models for an entertainment system. Determine the number of possible system configurations.
- 14. Education** A college student is preparing a course schedule for the next semester. The student must select one of two mathematics courses, one of three science courses, and one of five courses from the social sciences and humanities. How many schedules are possible?
- 15. Education** In how many ways can a 10-question true-false exam be answered? (Assume that no questions are omitted.)
- 16. Physiology** In a physiology class, a student must dissect three different specimens. The student can select one of nine earthworms, one of four frogs, and one of seven fetal pigs. In how many ways can the student select the specimens?
- 17. Fundamental Counting Principle** How many three-digit numbers can be formed under each condition?
 - The leading digit cannot be 0.
 - The leading digit cannot be 0 and no repetition of digits is allowed.
 - The leading digit cannot be 0 and the number must be a multiple of 5.

18. Fundamental Counting Principle How many four-digit numbers can be formed under each condition?

- The leading digit cannot be 0.
- The leading digit cannot be 0 and the number must be less than 5000.
- The leading digit cannot be 0 and the number must be even.

✓ **19. Telecommunication** In 2010, the state of Nevada had two area codes. Using the information about telephone numbers given in Example 4, how many telephone numbers could Nevada's phone system have accommodated?

20. Fundamental Counting Principle In Pennsylvania, each standard automobile license plate number consists of three letters followed by a four-digit number. How many distinct Pennsylvania license plate numbers can be formed?

21. Radio Broadcasting Typically radio stations are identified by four "call letters." Radio stations east of the Mississippi River have call letters that start with the letter W and radio stations west of the Mississippi River have call letters that start with the letter K.

- Find the number of different sets of radio station call letters that are possible in the United States.
- Find the number of different sets of radio station call letters that are possible when the call letters must include a Q.

22. Banking ATM personal identification number (PIN) codes typically consist of four-digit sequences of numbers.

- Find the total number of ATM codes possible.
- Find the total number of ATM codes possible when the first digit is not 0.

23. Operations Research In 1963, the United States Postal Service launched the Zoning Improvement Plan (ZIP) Code to streamline the mail-delivery system. A ZIP code consists of a five-digit sequence of numbers.

- (a) Find the number of ZIP codes consisting of five digits.
 (b) Find the number of ZIP codes consisting of five digits when the first digit is 1 or 2.

24. Operations Research In 1983, in order to identify small geographic segments, such as city blocks or a single building, within a delivery code, the post office began to use an expanded ZIP code called ZIP+4, which is composed of the original five-digit code plus a four-digit add-on code.

- (a) Find the number of ZIP codes consisting of five digits followed by the four additional digits.
 (b) Find the number of ZIP codes consisting of five digits followed by the four additional digits when the first number of the five-digit code is 1 or 2.

25. Finding Permutations Three couples have reserved seats in a row for a concert. In how many different ways can they be seated when (a) there are no seating restrictions? (b) each couple sits together?

26. Finding Permutations In how many orders can five girls and three boys walk through a doorway single file when (a) there are no restrictions? (b) the girls walk through before the boys?

Finding a Permutation In Exercises 27–32, evaluate ${}_nP_r$ using the formula from this section.

27. ${}_4P_4$ 28. ${}_5P_5$
 29. ${}_8P_3$ 30. ${}_{20}P_2$
 31. ${}_5P_4$ 32. ${}_7P_4$

Finding a Permutation Using a Graphing Utility In Exercises 33–36, evaluate ${}_nP_r$ using a graphing utility.

33. ${}_{20}P_6$ 34. ${}_{10}P_8$
 35. ${}_{120}P_4$ 36. ${}_{100}P_3$

✓ **37. Photography** In how many ways can five children posing for a photograph line up in a row?

38. Transportation Design In how many ways can six people sit in a six-passenger minivan?

39. Politics The nine justices of the U.S. Supreme Court pose for a photograph while standing in a straight line, as opposed to the typical pose of two rows. How many different orders of the justices are possible for this photograph?

40. Manufacturing Four processes are involved in assembling a product, and they can be performed in any order. The management wants to test each order to determine which is the least time-consuming. How many different orders will have to be tested?

✓ **41. Politics** From a pool of 12 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many ways can the offices be filled?

42. Physical Education How many different batting orders can a baseball coach create from a team of 15 players when there are nine positions to fill?

43. Combinatorics A school locker has a dial lock on which there are 37 numbers from 0 to 36. Find the total number of possible lock combinations when the lock requires a three-digit sequence of left-right-left and the numbers can be repeated.

44. Athletics Eight sprinters have qualified for the finals in the 100-meter dash at the NCAA national track meet. How many different orders of the top three finishes are possible? (Assume there are no ties.)

Finding Permutations In Exercises 45 and 46, use the letters A, B, C, and D.

45. Write all permutations of the letters.
 46. Write all permutations of the letters when the letters B and C must remain between the letters A and D.

Finding Distinguishable Permutations In Exercises 47–50, find the number of distinguishable permutations of the group of letters.

47. A, A, G, E, E, E, M 48. B, B, B, T, T, T, T, T
 ✓ 49. A, L, G, E, B, R, A 50. M, I, S, S, I, S, S, I, P, P, I

Finding a Combination In Exercises 51–56, evaluate ${}_nC_r$ using the formula from this section.

51. ${}_5C_2$ 52. ${}_6C_3$
 53. ${}_4C_1$ 54. ${}_5C_1$
 55. ${}_{25}C_0$ 56. ${}_{20}C_0$

Finding a Combination Using a Graphing Utility In Exercises 57–60, evaluate ${}_nC_r$ using a graphing utility.

57. ${}_{20}C_5$ 58. ${}_{10}C_7$
 59. ${}_{42}C_5$ 60. ${}_{50}C_6$

Finding a Combination In Exercises 61 and 62, use the letters A, B, C, D, E, and F.

✓ **61.** Write all possible selections of two letters that can be formed from the letters. (The order of the two letters is not important.)

62. Write all possible selections of three letters that can be formed from the letters. (The order of the three letters is not important.)

63. Politics As of March 2010, the U.S. Senate Committee on Indian Affairs had 15 members. Party affiliation is not a factor in selection. How many different committees are possible from the 100 U.S. senators?

64. Education You can answer any 12 questions from a total of 14 questions on an exam. In how many different ways can you select the questions?

65. **Why you should learn it** (p. 607) Powerball is played with 59 white balls, numbered 1 through 59, and 39 red balls, numbered 1 through 39. Five white balls and one red ball, the Powerball, are drawn. In how many ways can a player select the six numbers?



66. **Geometry** Three points that are not collinear determine three lines. How many lines are determined by nine points, no three of which are collinear?
67. **Quality Engineering** A shipment of 30 flat screen televisions contains three defective units. In how many ways can a vending company purchase four of these units and receive (a) all good units, (b) two good units, and (c) at least two good units?
68. **Game Theory** You are dealt five cards from an ordinary deck of 52 playing cards. In how many ways can you get a full house? (A full house consists of three of one kind and two of another. For example, 8-8-8-5-5 and K-K-K-10-10 are full houses.)
69. **Game Theory** Five cards are chosen from a standard deck of 52 cards. How many five-card combinations contain two jacks and three aces?
70. **Law** A law office interviews paralegals for 10 openings. There are 13 paralegals with two years of experience and 20 paralegals with one year of experience. How many combinations of seven paralegals with two years of experience and three paralegals with one year of experience are possible?
71. **Administration** A six-member research committee is to be formed having one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 25 students in contention for the committee. How many six-member committees are possible?
72. **Psychology** The number of possible interpersonal relationships increases dramatically as the size of a group increases. Determine the numbers of different two-person relationships that are possible in groups of people of sizes (a) 3, (b) 8, (c) 12, and (d) 20.

Geometry In Exercises 73–76, find the number of diagonals of the polygon. (A line segment connecting any two nonadjacent vertices is called a *diagonal* of a polygon.)

73. Pentagon
74. Hexagon
75. Octagon
76. Nonagon

Solving an Equation In Exercises 77–84, solve for n .

77. $14 \cdot {}_n P_3 = {}_{n+2} P_4$
78. ${}_n P_5 = 18 \cdot {}_{n-2} P_4$
79. ${}_n P_4 = 10 \cdot {}_{n-1} P_3$
80. ${}_n P_6 = 12 \cdot {}_{n-1} P_5$

81. ${}_{n+1} P_3 = 4 \cdot {}_n P_2$
82. ${}_{n+2} P_3 = 6 \cdot {}_{n+2} P_1$
83. $4 \cdot {}_{n+1} P_2 = {}_{n+2} P_3$
84. $5 \cdot {}_{n-1} P_1 = {}_n P_2$

Conclusions

True or False? In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.

85. The number of pairs of letters that can be formed from any of the first 13 letters in the alphabet (A–M), where repetitions are allowed, is an example of a permutation.
86. The number of permutations of n elements can be derived by using the Fundamental Counting Principle.
87. **Think About It** Can your calculator evaluate ${}_{100} P_{80}$? If not, explain why.

88. **CAPSTONE** Decide whether each scenario should be counted using permutations or combinations. Explain your reasoning. (Do not calculate.)

- (a) Number of ways 10 people can line up in a row for concert tickets
- (b) Number of different arrangements of three types of flowers from an array of 20 types
- (c) Number of four-digit pin numbers for a debit card
- (d) Number of two-scoop ice cream sundaes created from 31 different flavors

89. **Writing** Explain in your own words the meaning of ${}_n P_r$.
90. Without calculating the numbers, determine which of the following is greater. Explain.
- (a) The number of combinations of 10 elements taken six at a time
- (b) The number of permutations of 10 elements taken six at a time

Proof In Exercises 91–94, prove the identity.

91. ${}_n P_{n-1} = {}_n P_n$
92. ${}_n C_n = {}_n C_0$
93. ${}_n C_{n-1} = {}_n C_1$
94. ${}_n C_r = \frac{{}_n P_r}{r!}$

Cumulative Mixed Review

Solving an Equation In Exercises 95 and 96, solve the equation. Round your answer to three decimal places, if necessary.

95. $\log_2(x - 3) = 5$
96. $e^{x/3} = 16$

Using Cramer's Rule In Exercises 97 and 98, use Cramer's Rule to solve the system of equations.

97. $\begin{cases} -5x + 3y = -14 \\ 7x - 2y = 2 \end{cases}$
98. $\begin{cases} 8x + y = 35 \\ 6x + 2y = 10 \end{cases}$

8.6 Probability

The Probability of an Event

Any happening whose result is uncertain is called an **experiment**. The possible results of the experiment are **outcomes**, the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event**.

For instance, when a six-sided die is tossed, the sample space can be represented by the numbers 1 through 6. For the experiment to be fair, each of the outcomes is *equally likely*.

To describe a sample space in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

Example 1 Finding the Sample Space



Find the sample space for each of the following.

- One coin is tossed.
- Two coins are tossed.
- Three coins are tossed.

Solution

- Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space S is

$$S = \{H, T\}.$$

- Because either coin can land heads up or tails up, the possible outcomes are as follows.

HH = heads up on both coins

HT = heads up on first coin and tails up on second coin

TH = tails up on first coin and heads up on second coin

TT = tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Note that this list distinguishes between the two cases

HT and TH

even though these two outcomes appear to be similar.

- Following the notation in part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases

HHT , HTH , and THH

and among the cases

HTT , THT , and TTH .

 **CHECKPOINT** Now try Exercise 9.

What you should learn

- Find probabilities of events.
- Find probabilities of mutually exclusive events.
- Find probabilities of independent events.

Why you should learn it

You can use probability to solve a variety of problems that occur in real life. For instance, in Exercise 54 on page 623, you are asked to use probability to help analyze the age distribution of unemployed workers.



To calculate the probability of an event, count the number of outcomes in the event and in the sample space. The *number of equally likely outcomes* in event E is denoted by $n(E)$, and the number of equally likely outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is given by

$$P(E) = \frac{n(E)}{n(S)}.$$

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number from 0 to 1, inclusive. That is,

$$0 \leq P(E) \leq 1$$

as indicated in Figure 8.13. If $P(E) = 0$, then event E *cannot occur*, and E is called an **impossible event**. If $P(E) = 1$, then event E *must occur*, and E is called a **certain event**.

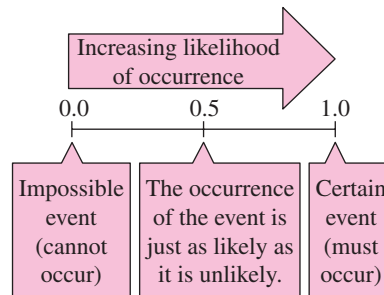


Figure 8.13

Example 2 Finding the Probability of an Event



- Two coins are tossed. What is the probability that both land heads up?
- A card is drawn at random from a standard deck of playing cards. What is the probability that it is an ace?

Solution

- Following the procedure in Example 1(b), let

$$E = \{HH\}$$

and

$$S = \{HH, HT, TH, TT\}.$$

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

- Because there are 52 cards in a standard deck of playing cards and there are four aces (one of each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

CHECKPOINT Now try Exercise 13.

Note that a probability can be written as a fraction, a decimal, or a percent. For instance, in Example 2(a), the probability of getting two heads can be written as

$$\frac{1}{4}, 0.25, \text{ or } 25\%.$$

Explore the Concept



Toss two coins 40 times and write down the number of heads that occur on each toss (0, 1, or 2). How many times did two heads occur? Without performing the experiment, how many times would you expect two heads to occur when two coins are tossed 1000 times?

You could have written out the sample space in Example 2(b) and simply counted the outcomes in the desired event. For larger sample spaces, however, using the counting principles discussed in Section 8.5 should save you time.

Example 3 Finding the Probability of an Event



Twelve-sided dice, as shown in Figure 8.14, can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die. Show that these dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of the various outcomes.

Solution

For an ordinary six-sided die, each of the numbers 1, 2, 3, 4, 5, and 6 occurs only once, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For a 12-sided die, each number occurs twice, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

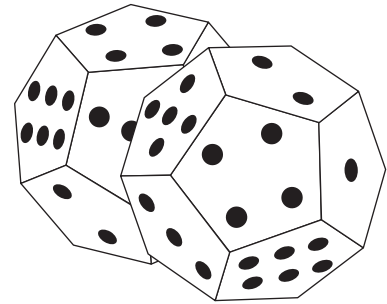


Figure 8.14

CHECKPOINT Now try Exercise 23.

Example 4 Random Selection



The numbers of colleges and universities in various regions of the United States in 2008 are shown in Figure 8.15. One institution is selected at random. What is the probability that the institution is in one of the three southern regions? (Source: U.S. National Center for Education Statistics)

Solution

From the figure, the total number of colleges and universities is 4347. Because there are

$$406 + 286 + 754 = 1446$$

colleges and universities in the three southern regions, the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1446}{4347} \approx 0.333.$$

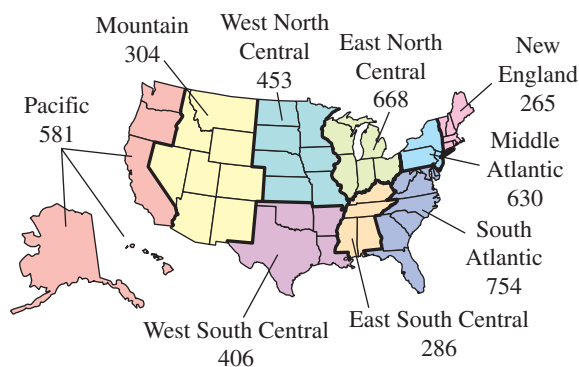


Figure 8.15

CHECKPOINT Now try Exercise 27.

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common. In the terminology of sets, the intersection of A and B is the empty set, which implies that

$$P(A \cap B) = 0.$$

For instance, when two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example 5 The Probability of a Union



One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution

Because the deck has 13 hearts, the probability of selecting a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts and face cards (see Figure 8.16), it follows that

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0.42. \end{aligned}$$

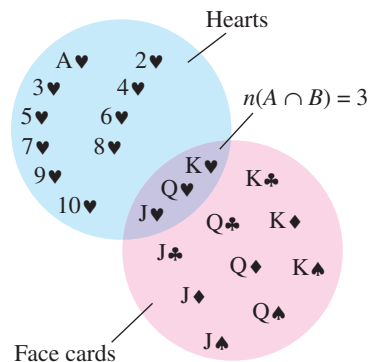



Figure 8.16

CHECKPOINT Now try Exercise 41.

Example 6 Probability of Mutually Exclusive Events

The human resources department of a company has compiled data on the numbers of employees who have been with the company for various periods of time. The results are shown in the table.



Years of service	Number of employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	37
35–39	21
40–44	8



Human Resources Manager

An employee is chosen at random. What is the probability that the employee has

- 4 or fewer years of service?
- 9 or fewer years of service?

Solution

- a. To begin, add the numbers of employees.

$$157 + 89 + 74 + 63 + 42 + 38 + 37 + 21 + 8 = 529$$

So, the total number of employees is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. From the table, you know there are 157 employees with 0 to 4 years of service. The probability of choosing an employee who has 4 or fewer years of service is

$$P(A) = \frac{157}{529} \approx 0.30.$$

- b. Let event B represent choosing an employee with 5 to 9 years of service. From the table, you know that there are 89 employees with 5 to 9 years of service. Then

$$P(B) = \frac{89}{529}.$$

Because event A from part (a) and event B have no outcomes in common, you can conclude that these two events are mutually exclusive and that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.47. \end{aligned}$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.47.

 **CHECKPOINT** Now try Exercise 47.

Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. (See Figure 8.17.) To find the probability that two independent events will occur, *multiply* the probabilities of each.

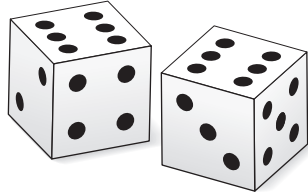


Figure 8.17

Probability of Independent Events

If A and B are **independent events**, then the probability that both A and B will occur is given by

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Example 7 Probability of Independent Events



A random number generator on a computer selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution

The probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}.$$

 **CHECKPOINT** Now try Exercise 49.

Example 8 Probability of Independent Events



In 2009, approximately 13% of the adult population of the United States got most of their news from the Internet. In a survey, 10 people were chosen at random from the adult population. What is the probability that all 10 got most of their news from the Internet? (Source: CBS News/New York Times Poll)

Solution

Let A represent choosing an adult who gets most of his or her news from the Internet. The probability of choosing an adult who got most of his or her news from the Internet is 0.13, the probability of choosing a second adult who got most of his or her news from the Internet is 0.13, and so on. Because these events are independent, you can conclude that the probability that all 10 people got most of their news from the Internet is

$$[P(A)]^{10} = (0.13)^{10} \approx 0.000000001.$$

 **CHECKPOINT** Now try Exercise 61.

8.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

- The set of all possible outcomes of an experiment is called the _____.
- To determine the probability of an event, use the formula $P(E) = \frac{n(E)}{n(S)}$, where $n(E)$ is _____ and $n(S)$ is _____.
- If two events from the same sample space have no outcomes in common, then the two events are _____.
- If the occurrence of one event has no effect on the occurrence of a second event, then the events are _____.
- Write an inequality that represents the possible values of the probability $P(E)$ of an event.
- What is the probability of an impossible event?
- What is the probability of a certain event?
- Match the probability formula with the correct probability name.

(a) Probability of the union of two events	(i) $P(A \cup B) = P(A) + P(B)$
(b) Probability of mutually exclusive events	(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(c) Probability of independent events	(iii) $P(A \text{ and } B) = P(A) \cdot P(B)$

Procedures and Problem Solving

Finding the Sample Space In Exercises 9–12, determine the sample space for the experiment.

- ✓ 9. A coin and a six-sided die are tossed.
10. A six-sided die is tossed twice and the results of roll 1 and roll 2 are recorded.
11. A taste tester has to rank three varieties of orange juice, A, B, and C, according to preference.
12. Two marbles are selected at random (without replacement) from a sack containing two red marbles, two blue marbles, and one yellow marble. The color of each marble is recorded.

Finding the Probability of an Event In Exercises 13–16, find the probability for the experiment of tossing a coin three times. Use the sample space

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

- ✓ 13. The probability of getting exactly two tails
14. The probability of getting a head on the first toss
15. The probability of getting at least one head
16. The probability of getting at least two heads

Finding the Probability of an Event In Exercises 17–20, find the probability for the experiment of selecting one card at random from a standard deck of 52 playing cards.

17. The card is a face card.
18. The card is a black card.

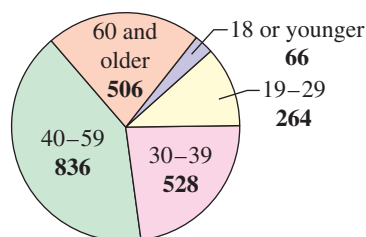
19. The card is a face card in the suit of spades.
20. The card is a numbered card (2–10).

Finding the Probability of an Event In Exercises 21–24, use the sample space from Exercise 10 to find the probability for the experiment of tossing a six-sided die twice.

21. The sum is 6.
22. The sum is at least 8.
- ✓ 23. The sum is less than 11.
24. The sum is odd or prime.

Random Selection In Exercises 25–28, one of a team's 2200 season ticket holders is selected at random to win a prize. The circle graph shows the ages of the season ticket holders. Find the probability of the event.

Ages of Season Ticket Holders



25. The winner is younger than 19 years old.
26. The winner is older than 39 years old.

- ✓ 27. The winner is 19 to 39 years old.
- 28. The winner is younger than 19 years old or older than 59 years old.

Using Combinations In Exercises 29–32, find the probability for the experiment of drawing two marbles at random (without replacement) from a bag containing one green, two yellow, and three red marbles. (*Hint: Use combinations to find the numbers of outcomes for the given event and sample space.*)

- 29. Both marbles are red.
- 30. Both marbles are yellow.
- 31. Neither marble is yellow.
- 32. The marbles are of different colors.

Finding the Probability of a Complement The *complement of an event A* is the collection of all outcomes in the sample space that are *not* in *A*. If the probability of *A* is $P(A)$, then the probability of the complement A' is given by $P(A') = 1 - P(A)$. In Exercises 33–36, you are given the probability that an event *will* happen. Find the probability that the event *will not* happen.

- 33. $P(E) = 0.75$
- 34. $P(E) = 0.\bar{2}$
- 35. $P(E) = \frac{2}{3}$
- 36. $P(E) = \frac{7}{8}$

Using the Probability of a Complement In Exercises 37–40, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.

- 37. $P(E') = 0.12$
- 38. $P(E') = 0.84$
- 39. $P(E') = \frac{13}{20}$
- 40. $P(E') = \frac{61}{100}$

The Probability of a Union In Exercises 41–44, one card is selected at random from a standard deck of 52 playing cards. Use a formula to find the probability of the union of the two events.

- ✓ 41. The card is a club or a king.
- 42. The card is a face card or a black card.
- 43. The card is a face card or a 2.
- 44. The card is a heart or a spade.

Probability of Mutually Exclusive Events In Exercises 45–48, use the table, which shows the age groups of students in a college sociology class.

Age	Number of students
18–19	11
20–21	18
22–30	2
31–40	1

A student from the class is randomly chosen for a project. Find the probability that the student is the given age.

- 45. 20 or 21 years old
- 46. 18 to 21 years old
- ✓ 47. Older than 21 years old
- 48. Younger than 31 years old

Probability of Independent Events In Exercises 49–52, a random number generator selects three numbers from 1 through 10. Find the probability of the event.

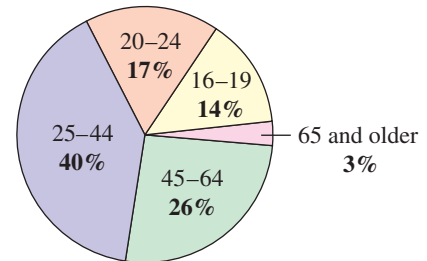
- ✓ 49. All three numbers are even.
- 50. All three numbers are less than or equal to 4.
- 51. Two numbers are less than 5 and the other number is 10.
- 52. One number is 2, 4, or 6, and the other two numbers are odd.

53. Political Science Taylor, Moore, and Perez are candidates for public office. It is estimated that Moore and Perez have about the same probability of winning, and Taylor is believed to be twice as likely to win as either of the others. Find the probability of each candidate’s winning the election.

54. Why you should learn it (p. 616) In 2008, there were approximately 8.92 million unemployed workers in the United States. The circle graph shows the age profile of these unemployed workers. (Source: U.S. Bureau of Labor Statistics)



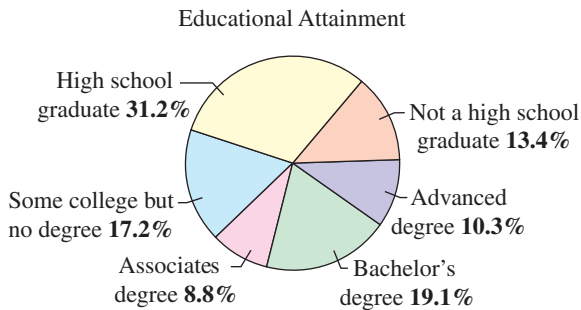
Ages of Unemployed Workers



- (a) Estimate the number of unemployed workers in the 16–19 age group.
- (b) What is the probability that a person selected at random from the population of unemployed workers is in the 25–44 age group?
- (c) What is the probability that a person selected at random from the population of unemployed workers is 25 to 64 years old?
- (d) What is the probability that a person selected at random from the population of unemployed workers is 45 or older?

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55. Education The levels of educational attainment of the United States population age 25 years or older in 2008 are shown in the circle graph. Use the fact that the population of people 25 years or older was 196.31 million in 2008. (Source: U.S. Census Bureau)



- (a) Estimate the number of people 25 years old or older who have advanced degrees.
- (b) Find the probability that a person 25 years old or older selected at random has earned a Bachelor's degree or higher.
- (c) Find the probability that a person 25 years old or older selected at random has earned a high school diploma or gone on to post-secondary education.

56. Political Science One hundred college students were interviewed to determine their political party affiliations and whether they favored a balanced budget amendment to the Constitution. The results of the study are listed in the table, where *D* represents Democrat and *R* represents Republican.



	<i>D</i>	<i>R</i>	Total
Favor	23	32	55
Oppose	25	9	34
Unsure	7	4	11
Total	55	45	100

A person is selected at random from the sample. Find the probability that the person selected is (a) a person who doesn't favor the amendment, (b) a Republican, and (c) a Democrat who favors the amendment.

In Exercises 57–60, the sample spaces are large and you should use the counting principles discussed in Section 8.5.

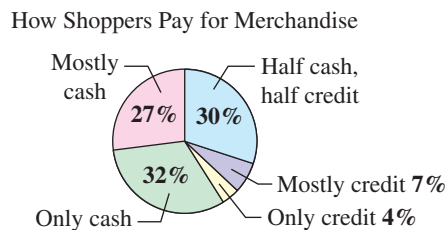
57. Using Counting Principles On a game show you are given five digits to arrange in the proper order to form the price of a car. If you are correct, you win the car. What is the probability of winning if you (a) guess the position of each digit and (b) know the first digit and guess the others?

58. Using Counting Principles The deck for a card game is made up of 108 cards. Twenty-five each are red, yellow, blue, and green, and eight are wild cards. Each player is randomly dealt a seven-card hand. What is the probability that a hand will contain (a) exactly two wild cards, and (b) two wild cards, two red cards, and three blue cards?

59. Using Counting Principles A shipment of 12 microwave ovens contains three defective units. A vending company has ordered four of these units, and because all are packaged identically, the selection will be random. What is the probability that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?

60. Using Counting Principles Two integers from 1 through 40 are chosen by a random number generator. What is the probability that (a) the numbers are both even, (b) one number is even and one is odd, (c) both numbers are less than 30, and (d) the same number is chosen twice?

61. Marketing Suppose that the methods used by shoppers to pay for merchandise are as shown in the circle graph. Two shoppers are chosen at random. What is the probability that both shoppers paid for their purchases only in cash?

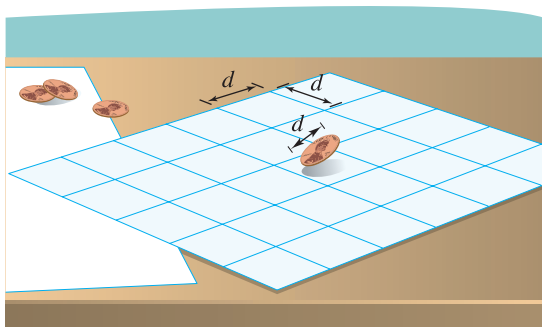


62. Aerospace Engineering A space vehicle has an independent backup system for one of its communication networks. The probability that either system will function satisfactorily for the duration of a flight is 0.985. What is the probability that during a given flight (a) both systems function satisfactorily, (b) at least one system functions satisfactorily, and (c) both systems fail?

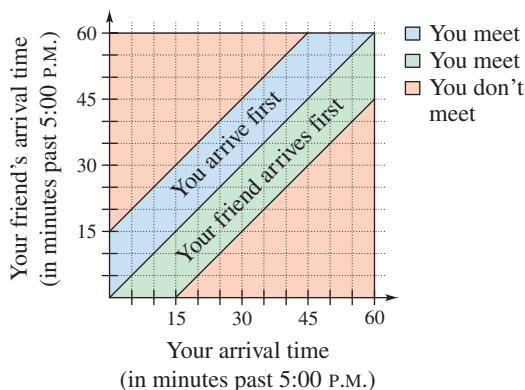
63. Marketing A sales representative makes sales on approximately one-fifth of all calls. On a given day, the representative contacts six potential clients. What is the probability that a sale will be made with (a) all six contacts, (b) none of the contacts, and (c) at least one contact?

64. Genetics Assume that the probability of the birth of a child of a particular sex is 50%. In a family with four children, what is the probability that (a) all the children are boys, (b) all the children are the same sex, and (c) there is at least one boy?

65. **Estimating π** A coin of diameter d is dropped onto a paper that contains a grid of squares d units on a side. (See figure.)
- (a) Find the probability that the coin covers a vertex of one of the squares on the grid.
 - (b) Perform the experiment 100 times and use the results to approximate π .



66. **Geometry** You and a friend agree to meet at your favorite fast food restaurant between 5:00 and 6:00 P.M. The one who arrives first will wait 15 minutes for the other, after which the first person will leave (see figure). What is the probability that the two of you will actually meet, assuming that your arrival times are random within the hour?



Conclusions

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

- 67. If the probability of an outcome in a sample space is 1, then the probability of the other outcomes in the sample space is 0.
- 68. If A and B are independent events with nonzero probabilities, then A can occur when B occurs.
- 69. **Exploration** Consider a group of n people.

- (a) Explain why the following pattern gives the probability that the n people have distinct birthdays.

$$n = 2: \frac{365}{365} \cdot \frac{364}{365} = \frac{365 \cdot 364}{365^2}$$

$$n = 3: \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = \frac{365 \cdot 364 \cdot 363}{365^3}$$

- (b) Use the pattern in part (a) to write an expression for the probability that four people ($n = 4$) have distinct birthdays.
- (c) Let P_n be the probability that the n people have distinct birthdays. Verify that this probability can be obtained recursively by

$$P_1 = 1 \quad \text{and} \quad P_n = \frac{365 - (n - 1)}{365} P_{n-1}.$$

- (d) Explain why $Q_n = 1 - P_n$ gives the probability that at least two people in a group of n people have the same birthday.
- (e) Use the results of parts (c) and (d) to complete the table.

n	10	15	20	23	30	40	50
P_n							
Q_n							

- (f) How many people must be in a group so that the probability of at least two of them having the same birthday is greater than $\frac{1}{2}$? Explain.

- 70. **Writing** Write a paragraph describing in your own words the difference between mutually exclusive events and independent events.
- 71. **Think About It** Let A and B be two events from the same sample space such that $P(A) = 0.76$ and $P(B) = 0.58$.
 - (a) Is it possible that A and B are mutually exclusive? Explain. Draw a diagram to support your answer.
 - (b) Is it possible that A' and B' are mutually exclusive? Explain. Draw a diagram to support your answer.
 - (c) Determine the possible range of $P(A \cup B)$.

72. **CAPSTONE** Give an example of each of the following events for the experiment of selecting one or more cards at random from a standard deck of 52 playing cards.

- (a) A union of two mutually exclusive events
- (b) A union of two events that are not mutually exclusive
- (c) Independent events
- (d) The complement of an event

Cumulative Mixed Review

Finding a Combination In Exercises 73–76, evaluate ${}_n C_r$. Verify your result using a graphing utility.

- 73. ${}_6 C_2$
- 74. ${}_9 C_5$
- 75. ${}_{11} C_8$
- 76. ${}_{16} C_{13}$

8 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
8.1	Use sequence notation to write the terms of sequences (p. 570).	$a_n = 7n - 4$; $a_1 = 7(1) - 4 = 3$, $a_2 = 7(2) - 4 = 10$, $a_3 = 7(3) - 4 = 17$, $a_4 = 7(4) - 4 = 24$	1–8
	Use factorial notation (p. 573).	If n is a positive integer, then $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$.	9–12
	Use summation notation to write sums (p. 574).	The sum of the first n terms of a sequence is represented by $\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n.$	13–20
	Find sums of infinite series (p. 575).	$\sum_{i=1}^{\infty} \frac{5}{10^i} = \frac{5}{10^1} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \frac{5}{10^5} + \cdots$ $= 0.5 + 0.05 + 0.005 + 0.0005 + 0.00005 + \cdots$ $= 0.55555 \dots$ $= \frac{5}{9}$	21–24
	Use sequences and series to model and solve real-life problems (p. 576).	A sequence can be used to model the resident population of the United States from 1980 through 2008. (See Example 10.)	25, 26
8.2	Recognize, write, and find the n th terms of arithmetic sequences (p. 581).	$a_n = 9n + 5$; $a_1 = 9(1) + 5 = 14$, $a_2 = 9(2) + 5 = 23$, $a_3 = 9(3) + 5 = 32$, $a_4 = 9(4) + 5 = 41$	27–38
	Find n th partial sums of arithmetic sequences (p. 584).	The sum of a finite arithmetic sequence with n terms is $S_n = \frac{n}{2}(a_1 + a_n).$	39–42
	Use arithmetic sequences to model and solve real-life problems (p. 585).	An arithmetic sequence can be used to find the total sales of a small business. (See Example 7.)	43, 44
8.3	Recognize, write, and find the n th terms of geometric sequences (p. 589).	$a_n = 3(4^n)$; $a_1 = 3(4^1) = 12$, $a_2 = 3(4^2) = 48$, $a_3 = 3(4^3) = 192$, $a_4 = 3(4^4) = 768$	45–56
	Find n th partial sums of geometric sequences (p. 592).	The sum of the finite geometric sequence $a_1, a_1r, a_1r^2, \dots, a_1r^{n-1}$ with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^n a_1r^{i-1} = a_1 \left(\frac{1-r^n}{1-r} \right).$	57–60
	Find sums of infinite geometric series (p. 593).	If $ r < 1$, then the infinite geometric series $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} + \cdots$ has the sum $S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1-r}.$	61–64
	Use geometric sequences to model and solve real-life problems (p. 594).	A finite geometric sequence can be used to find the balance in an annuity at the end of 2 years. (See Example 8.)	65, 66

	What did you learn?	Explanation and Examples	Review Exercises
8.4	Use the Binomial Theorem to calculate binomial coefficients (p. 599).	The Binomial Theorem: In the expansion of $(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_n C_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$, the coefficient of $x^{n-r}y^r$ is ${}_n C_r = \frac{n!}{(n-r)!r!}$	67–70
	Use binomial coefficients to write binomial expansions (p. 601).	$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$ $(x - 1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$	71–74
	Use Pascal's Triangle to calculate binomial coefficients (p. 603).	First several rows of Pascal's triangle: $\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & 1 & & 2 & & 1 & \\ & & 1 & & 3 & & 3 & & 1 \\ 1 & & 1 & & 4 & & 6 & & 4 & & 1 \end{array}$	75–78
8.5	Solve simple counting problems (p. 607).	A computer randomly generates an integer from 1 through 15. The computer can generate an integer that is divisible by 3 in 5 ways (3, 6, 9, 12, and 15).	79, 80
	Use the Fundamental Counting Principle to solve more complicated counting problems (p. 608).	Fundamental Counting Principle: Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.	81, 82
	Use permutations to solve counting problems (p. 609).	A permutation of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on. The number of permutations of n elements is given by $n \cdot (n - 1) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!$ In other words, there are $n!$ different ways that n elements can be ordered. The number of permutations of n elements taken r at a time is ${}_n P_r = \frac{n!}{(n-r)!}$	83–88, 91
	Use combinations to solve counting problems (p. 612).	The number of combinations of n elements taken r at a time is ${}_n C_r = \frac{n!}{(n-r)!r!}$	89, 90, 92
8.6	Find probabilities of events (p. 616).	If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the probability of event E is $P(E) = \frac{n(E)}{n(S)}$	93, 94
	Find probabilities of mutually exclusive events (p. 619).	If A and B are events in the same sample space, then the probability of A or B occurring is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$	95
	Find probabilities of independent events (p. 621).	If A and B are independent events, then the probability that both A and B will occur is $P(A \text{ and } B) = P(A) \cdot P(B)$.	96

8 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

8.1

Writing the Terms of a Sequence In Exercises 1 and 2, write the first five terms of the sequence. (Assume n begins with 1.)

$$1. a_n = \frac{2^n}{2^n + 1} \quad 2. a_n = \frac{(-1)^n}{n!}$$

Finding the n th Term of a Sequence In Exercises 3–6, write an expression for the *apparent* n th term of the sequence. (Assume n begins with 1.)

$$3. 5, 10, 15, 20, 25, \dots \quad 4. 50, 48, 46, 44, 42, \dots$$

$$5. 2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \dots \quad 6. \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \dots$$

A Recursive Sequence In Exercises 7 and 8, write the first five terms of the sequence defined recursively.

$$7. a_1 = 9, a_{k+1} = a_k - 4 \quad 8. a_1 = 49, a_{k+1} = a_k + 6$$

Simplifying Factorial Expressions In Exercises 9–12, simplify the factorial expression.

$$9. \frac{18!}{20!} \quad 10. \frac{10!}{8!}$$

$$11. \frac{(n+1)!}{(n-1)!} \quad 12. \frac{2n!}{(n+1)!}$$

Sigma Notation for Sums In Exercises 13–16, find the sum.

$$13. \sum_{i=1}^6 5 \quad 14. \sum_{k=2}^5 4k$$

$$15. \sum_{j=1}^4 \frac{6}{j^2} \quad 16. \sum_{i=1}^8 \frac{i}{i+1}$$

Writing a Sum Using Sigma Notation In Exercises 17–20, use sigma notation to write the sum. Then use a graphing utility to find the sum.

$$17. \frac{1}{2(1)} + \frac{1}{2(2)} + \frac{1}{2(3)} + \dots + \frac{1}{2(20)}$$

$$18. 2(1^2) + 2(2^2) + 2(3^2) + \dots + 2(9^2)$$

$$19. \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{9}{10}$$

$$20. 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots$$

Finding the Sum of a Series In Exercises 21–24, find (a) the fourth partial sum and (b) the sum of the infinite series.

$$21. \sum_{k=1}^{\infty} \frac{5}{10^k} \quad 22. \sum_{k=1}^{\infty} \frac{3}{2^k}$$

$$23. \sum_{k=1}^{\infty} 2(0.5)^k \quad 24. \sum_{k=1}^{\infty} 4(0.25)^k$$

25. Compound Interest A deposit of \$3000 is made in an account that earns 2% interest compounded quarterly. The balance in the account after n quarters is given by

$$a_n = 3000 \left(1 + \frac{0.02}{4} \right)^n, \quad n = 1, 2, 3, \dots$$

- (a) Compute the first eight terms of this sequence.
 (b) Find the balance in this account after 10 years by computing the 40th term of the sequence.

26. Education The average salaries a_n (in thousands of dollars) of college faculty members in the United States from 2001 through 2009 can be approximated by the model

$$a_n = 0.065n^2 + 1.43n + 58.2, \quad n = 1, 2, 3, \dots, 9$$

where n is the year, with $n = 1$ corresponding to 2001. (Source: American Association of University Professors)

- (a) Find the terms of this finite sequence for the given values of n .
 (b) Use a graphing utility to graph the sequence for the given values of n .
 (c) Use the graphing utility to construct a bar graph of the sequence for the given values of n .
 (d) Use the sequence to predict the average salaries of college faculty members for the years 2010 through 2016. Do your results seem reasonable? Explain.

8.2

Determining Whether a Sequence Is Arithmetic In Exercises 27–30, determine whether or not the sequence is arithmetic. If it is, find the common difference.

$$27. 5, 3, 1, -1, -3, \dots \quad 28. 0, 1, 3, 6, 10, \dots$$

$$29. \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \quad 30. \frac{9}{9}, \frac{8}{9}, \frac{7}{9}, \frac{6}{9}, \frac{5}{9}, \dots$$

Writing the Terms of an Arithmetic Sequence In Exercises 31–34, write the first five terms of the arithmetic sequence. Use the *table* feature of a graphing utility to verify your results.

$$31. a_1 = 3, d = 4 \quad 32. a_1 = 8, d = -2$$

$$33. a_4 = 10, a_{10} = 28 \quad 34. a_2 = 14, a_6 = 22$$

Writing the Terms of an Arithmetic Sequence In Exercises 35 and 36, write the first five terms of the arithmetic sequence. Find the common difference and write the n th term of the sequence as a function of n .

$$35. a_1 = 35, a_{k+1} = a_k - 3 \quad 36. a_1 = 15, a_{k+1} = a_k + \frac{5}{2}$$

Finding the n th Term of an Arithmetic Sequence In Exercises 37 and 38, find a formula for a_n for the arithmetic sequence and find the sum of the first 25 terms of the sequence.

37. $a_1 = 100, d = -3$ 38. $a_1 = 10, a_3 = 28$

Finding a Partial Sum of an Arithmetic Sequence In Exercises 39 and 40, find the partial sum. Use a graphing utility to verify your result.

39. $\sum_{j=1}^{10} (2j - 3)$ 40. $\sum_{k=1}^{11} (\frac{2}{3}k + 4)$

41. Find the sum of the first 50 positive multiples of 5.
 42. Find the sum of the integers from 10 to 70.
 43. **Accounting** The starting salary for an accountant is \$36,000 with a guaranteed salary increase of \$2250 per year. Determine (a) the salary during the fifth year and (b) the total compensation through 5 full years of employment.
 44. **Agriculture** In his first trip around a field, a farmer makes 123 bales. In his second trip, he makes 11 fewer bales. Because each trip is shorter than the preceding trip, the farmer estimates that the same pattern will continue. Estimate the total number of bales made if there are another five trips around the field.

8.3

Identifying a Geometric Sequence In Exercises 45–48, determine whether or not the sequence is geometric. If it is, find the common ratio.

45. 5, 10, 20, 40, . . . 46. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$
 47. 54, -18, 6, -2, . . . 48. $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \dots$

Finding the Terms of a Geometric Sequence In Exercises 49–52, write the first five terms of the geometric sequence.

49. $a_1 = 4, r = -\frac{1}{4}$ 50. $a_1 = 2, r = \frac{3}{2}$
 51. $a_1 = 9, a_3 = 4$ 52. $a_1 = 2, a_3 = 12$

Finding the n th Term of a Geometric Sequence In Exercises 53 and 54, write the first five terms of the geometric sequence. Find the common ratio and write the n th term of the sequence as a function of n .

53. $a_1 = 120, a_{k+1} = \frac{1}{3}a_k$ 54. $a_1 = 200, a_{k+1} = 0.1a_k$

Finding a Term of a Geometric Sequence In Exercises 55 and 56, find the indicated term of the geometric sequence (a) algebraically and (b) using the *table* feature of a graphing utility.

55. $a_1 = 16, a_2 = -8$, 6th term
 56. $a_3 = 6, a_4 = 1$, 9th term

Finding the Sum of a Finite Geometric Sequence In Exercises 57–60, find the sum. Use a graphing utility to verify your result.

57. $\sum_{i=1}^7 2^{i-1}$ 58. $\sum_{i=1}^5 3^{i-1}$
 59. $\sum_{n=1}^7 (-4)^{n-1}$ 60. $\sum_{n=1}^4 12(-\frac{1}{2})^{n-1}$

Finding the Sum of an Infinite Geometric Series In Exercises 61–64, find the sum of the infinite geometric series.

61. $\sum_{i=1}^{\infty} 4(\frac{7}{8})^{i-1}$ 62. $\sum_{i=1}^{\infty} 6(\frac{1}{3})^{i-1}$
 63. $\sum_{k=1}^{\infty} 4(\frac{2}{3})^{k-1}$ 64. $\sum_{k=1}^{\infty} 1.3(\frac{1}{10})^{k-1}$

65. **Finance** A company buys a fleet of six vans for \$130,000. During the next 5 years, the fleet will depreciate at a rate of 30% per year. (That is, at the end of each year, the depreciated value will be 70% of the value at the beginning of the year.)
 (a) Find the formula for the n th term of a geometric sequence that gives the value of the fleet t full years after it was purchased.
 (b) Find the depreciated value of the fleet at the end of 5 full years.
 66. **Finance** A deposit of \$80 is made at the beginning of each month in an account that pays 4% interest, compounded monthly. The balance A in the account at the end of 4 years is given by

$$A = 80\left(1 + \frac{0.04}{12}\right)^1 + \dots + 80\left(1 + \frac{0.04}{12}\right)^{48}.$$

Find A .

8.4

Finding Binomial Coefficients In Exercises 67–70, find the binomial coefficient. Use a graphing utility to verify your result.

67. ${}_{10}C_8$ 68. ${}_{12}C_5$
 69. ${}_9C_4$ 70. ${}_{14}C_{12}$

Expanding a Binomial In Exercises 71–74, use the Binomial Theorem to expand and simplify the expression.

71. $(x + 5)^4$ 72. $(y - 3)^3$
 73. $(a - 4b)^5$ 74. $(3x + y)^7$

Using Pascal's Triangle In Exercises 75–78, use Pascal's Triangle to find the binomial coefficient.

75. ${}_6C_3$ 76. ${}_9C_7$
 77. ${}_8C_4$ 78. ${}_{10}C_5$

8.5

- 79. Probability** Slips of paper numbered 1 through 15 are placed in a hat. In how many ways can two numbers be drawn so that the sum of the numbers is 12? Assume the random selection is without replacement.
- 80. Aviation** Ten people are boarding an aircraft. Two have tickets for first class and board before those in economy class. In how many ways can the ten people board the aircraft?
- 81. Education** A college student is preparing a course schedule of four classes for the next semester. The student can choose from the open sections shown in the table.



Course	Sections
Math 100	001–004
Economics 110	001–003
English 105	001–006
Humanities 101	001–003

- (a) Find the number of possible schedules that the student can create from the offerings.
- (b) Find the number of possible schedules that the student can create from the offerings when two of the Math 100 sections are closed.
- (c) Find the number of possible schedules that the student can create from the offerings when two of the Math 100 sections and four of the English 105 sections are closed.
- 82. Marketing** A telemarketing firm is making calls to prospective customers by randomly dialing seven-digit phone numbers within an area code.
- (a) Find the number of possible calls that the telemarketer can make.
- (b) The telemarketing firm is calling only within exchanges that begin with a “7” or a “6.” How many different calls are possible?
- (c) The telemarketing firm is calling only within exchanges that do not begin with a “0” or a “1.” How many different calls are possible?

Finding a Permutation In Exercises 83 and 84, evaluate ${}_n P_r$ using the formula from Section 8.5. Use a graphing utility to verify your result.

83. ${}_{12}P_{10}$ **84.** ${}_6P_4$

Finding Distinguishable Permutations In Exercises 85 and 86, find the number of distinguishable permutations of the group of letters.

85. C, A, L, C, U, L, U, S **86.** I, N, T, E, G, R, A, T, E

Solving an Equation In Exercises 87 and 88, solve for n .

87. ${}_{n+1}P_2 = 4 \cdot {}_n P_1$ **88.** $8 \cdot {}_n P_2 = {}_{n+1}P_3$

Finding a Combination In Exercises 89 and 90, evaluate ${}_n C_r$ using the formula from Section 8.5. Use a graphing utility to verify your result.

89. ${}_8 C_6$ **90.** ${}_{50} C_{48}$

- 91. Athletics** There are 12 bicyclists entered in a race. In how many different orders could the 12 bicyclists finish? (Assume there are no ties.)
- 92. Athletics** From a pool of seven juniors and eleven seniors, four co-captains will be chosen for the football team. How many different combinations are possible if two juniors and two seniors are to be chosen?

8.6

- 93. Fashion** A man has five pairs of socks (no two pairs are the same color). He randomly selects two socks from a drawer. What is the probability that he gets a matched pair?
- 94. Library Science** A child returns a five-volume set of books to a bookshelf. The child is not able to read, and so cannot distinguish one volume from another. What is the probability that the books are shelved in the correct order?
- 95. Education** A sample of college students, faculty members, and administrators were asked whether they favored a proposed increase in the annual activity fee to enhance student life on campus. The results of the study are shown in the table.



	Favor	Oppose	Total
Students	237	163	400
Faculty	37	38	75
Admin.	18	7	25
Total	292	208	500

A person is selected at random from the sample. Find each probability.

- (a) The person is not in favor of the proposal.
- (b) The person is a student.
- (c) The person is a faculty member and is in favor of the proposal.
- 96. Probability** A six-sided die is rolled six times. What is the probability that each side appears exactly once?

Conclusions

True or False? In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.

97. $\frac{(n+2)!}{n!} = (n+2)(n+1)$ **98.** $\sum_{k=1}^8 3k = 3 \sum_{k=1}^8 k$

- 99. Writing** In your own words, explain what makes a sequence (a) arithmetic and (b) geometric.

8 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

In Exercises 1–4, write the first five terms of the sequence.

1. $a_n = \left(-\frac{2}{3}\right)^{n-1}$ (Begin with $n = 1$.)
2. $a_1 = 12$ and $a_{k+1} = a_k + 4$
3. $b_n = \frac{(-1)^n x^n}{n}$ (Begin with $n = 1$.)
4. $b_n = \frac{(-1)^{2n+1} x^{2n+1}}{(2n+1)!}$ (Begin with $n = 1$.)
5. Simplify $\frac{11! \cdot 4!}{4! \cdot 7!}$.
6. Simplify $\frac{n!}{(n+1)!}$.
7. Simplify $\frac{2n!}{(n-1)!}$.
8. Write an expression for the *apparent* n th term of the sequence
2, 5, 10, 17, 26, . . . (Assume n begins with 1.)

In Exercises 9 and 10, find a formula for the n th term of the sequence.

9. Arithmetic: $a_1 = 5000$, $d = -100$
10. Geometric: $a_1 = 4$, $a_{k+1} = \frac{1}{2}a_k$
11. Use sigma notation to write $\frac{2}{3(1)+1} + \frac{2}{3(2)+1} + \cdots + \frac{2}{3(12)+1}$.
12. Use sigma notation to write $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$.

In Exercises 13–15, find the sum.

13. $\sum_{n=1}^7 (8n - 5)$
14. $\sum_{n=1}^8 24\left(\frac{1}{6}\right)^{n-1}$
15. $5 - 2 + \frac{4}{5} - \frac{8}{25} + \frac{16}{125} - \cdots$
16. Use the Binomial Theorem to expand and simplify $(2a - 5b)^4$.

In Exercises 17–20, evaluate the expression.

17. ${}_9C_3$
18. ${}_{20}C_3$
19. ${}_9P_2$
20. ${}_{70}P_3$
21. Solve for n in $4 \cdot {}_n P_3 = {}_{n+1} P_4$.
22. How many distinct license plates can be issued consisting of one letter followed by a three-digit number?
23. Four students are randomly selected from a class of 25 to answer questions from a reading assignment. In how many ways can the four be selected?
24. A card is drawn from a standard deck of 52 playing cards. Find the probability that it is a red face card.
25. In 2010, five of the eleven men's basketball teams in the Big Ten Conference participated in the NCAA Men's Basketball Championship Tournament. What is the probability that a random selection of five teams from the Big Ten Conference will be the same as the five teams that participated in the tournament?
26. Two integers from 1 to 60 are chosen by a random number generator. What is the probability that (a) both numbers are odd, (b) both numbers are less than 12, and (c) the same number is chosen twice?
27. A weather forecast indicates that the probability of snow is 75%. What is the probability that it will not snow?

Proofs in Mathematics

Properties of Sums (p. 574)

$$1. \sum_{i=1}^n c = cn, \quad c \text{ is a constant.}$$

$$2. \sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i, \quad c \text{ is a constant.}$$

$$3. \sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$4. \sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

Proof

Each of these properties follows directly from the properties of real numbers.

$$1. \sum_{i=1}^n c = c + c + c + \cdots + c = cn \quad n \text{ terms}$$

The Distributive Property is used in the proof of Property 2.

$$\begin{aligned} 2. \sum_{i=1}^n ca_i &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= c \sum_{i=1}^n a_i \end{aligned}$$

The proof of Property 3 uses the Commutative and Associative Properties of Addition.

$$\begin{aligned} 3. \sum_{i=1}^n (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i \end{aligned}$$

The proof of Property 4 uses the Commutative and Associative Properties of Addition and the Distributive Property.

$$\begin{aligned} 4. \sum_{i=1}^n (a_i - b_i) &= (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdots + (a_n - b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (-b_1 - b_2 - b_3 - \cdots - b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) - (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^n a_i - \sum_{i=1}^n b_i \end{aligned}$$

Infinite Series

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity; throughout the next quarter of the interval at double the intensity; throughout the following eighth at triple the intensity and so ad infinitum; The average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots$$

is 2.

The Sum of a Finite Arithmetic Sequence (p. 584)

The sum of a finite arithmetic sequence with n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Proof

Begin by generating the terms of the arithmetic sequence in two ways. In the first way, repeatedly add d to the first term to obtain

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n \\ &= a_1 + [a_1 + d] + [a_1 + 2d] + \cdots + [a_1 + (n - 1)d]. \end{aligned}$$

In the second way, repeatedly subtract d from the n th term to obtain

$$\begin{aligned} S_n &= a_n + a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1 \\ &= a_n + [a_n - d] + [a_n - 2d] + \cdots + [a_n - (n - 1)d]. \end{aligned}$$

When you add these two versions of S_n , the multiples of d subtract out and you obtain

$$\begin{aligned} 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) && n \text{ terms} \\ 2S_n &= n(a_1 + a_n) \\ S_n &= \frac{n}{2}(a_1 + a_n). \end{aligned}$$

The Sum of a Finite Geometric Sequence (p. 592)

The sum of the finite geometric sequence

$$a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \dots, a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^n a_1r^{i-1} = a_1 \left(\frac{1 - r^n}{1 - r} \right).$$

Proof

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} \\ rS_n &= a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n && \text{Multiply by } r. \end{aligned}$$

Subtracting the second equation from the first yields

$$S_n - rS_n = a_1 - a_1r^n.$$

So, $S_n(1 - r) = a_1(1 - r^n)$, and, because $r \neq 1$, you have

$$S_n = a_1 \left(\frac{1 - r^n}{1 - r} \right).$$

The Binomial Theorem (p. 599)In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_n C_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_n C_r = \frac{n!}{(n-r)!r!}.$$

Proof

The Binomial Theorem can be proved quite nicely using **mathematical induction**. (For information about mathematical induction, see Appendix G on this textbook's *Companion Website*.) The steps are straightforward but look a little messy, so only an outline of the proof is presented.

1. For $n = 1$, you have

$$(x + y)^1 = x^1 + y^1 = {}_1 C_0 x + {}_1 C_1 y$$

and the formula is valid.

2. Assuming that the formula is true for $n = k$, the coefficient of $x^{k-r}y^r$ is

$${}_k C_r = \frac{k!}{(k-r)!r!} = \frac{k(k-1)(k-2)\cdots(k-r+1)}{r!}.$$

To show that the formula is true for $n = k + 1$, look at the coefficient of $x^{k+1-r}y^r$ in the expansion of

$$(x + y)^{k+1} = (x + y)^k(x + y).$$

From the right-hand side, you can determine that the term involving $x^{k+1-r}y^r$ is the sum of two products.

$$\begin{aligned} &({}_k C_r x^{k-r}y^r)(x) + ({}_k C_{r-1} x^{k+1-r}y^{r-1})(y) \\ &= \left[\frac{k!}{(k-r)!r!} + \frac{k!}{(k+1-r)!(r-1)!} \right] x^{k+1-r}y^r \\ &= \left[\frac{(k+1-r)k!}{(k+1-r)!r!} + \frac{k!r}{(k+1-r)!r!} \right] x^{k+1-r}y^r \\ &= \left[\frac{k!(k+1-r+r)}{(k+1-r)!r!} \right] x^{k+1-r}y^r \\ &= \left[\frac{(k+1)!}{(k+1-r)!r!} \right] x^{k+1-r}y^r \\ &= {}_{k+1} C_r x^{k+1-r}y^r \end{aligned}$$

So, by mathematical induction, the Binomial Theorem is valid for all positive integers n .

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n . If

1. P_1 is true, and
2. the truth of P_k implies the truth of P_{k+1} for every positive integer k ,

then P_n must be true for all positive integers n .